ELLIPTIC FORMAL GROUP LAWS, INTEGRAL HIRZEBRUCH GENERA AND KRICHEVER GENERA.

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Introduction.

The theory of formal group laws plays an important role in many directions of modern mathematics. In the classical papers and wellknown textbooks (see [34], [14], [22], [30]) we can find fundamental results concerning the group structure on elliptic curves. This results lead to remarkable formal group laws. We consider an elliptic curve as an irreducible non-singular projective algebraic curve of genus 1 furnished with a point 0, the zero of the group law. Any such curve has a plane cubic model of the form

$$y^2 + \mu_1 xy + \mu_3 y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6.$$

In Tate coordinates the geometrical addition laws on this curves correspond to the general formal group law over the ring $\mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$. We study the structure of this law and the differential equation that determines its exponent. We describe a 5-parametric family of Hirzebruch genera with integer values on stably complex manifolds. We introduce the general Krichever genus, which is given by a generalized Baker-Akhiezer function. This function has many of the fundamental properties of the Baker-Akhiezer function, but unlike it, it is not meromorphic, because it can have two branch points in the parallelogram of periods.

In section 1 we give necessary facts on Hurwitz series, formal group laws and Hirzebruch genera and formulate the problems connecting this three fundamental objects. The main part of the paper contains the results on this problems in the cases when this objects are defined by the elliptic curves.

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1. Main notions and problems.

Let A be a commutative associative evenly graded torsion-free ring $A = \sum_{k\geq 0} A_{-2k}$ with identity element $1 \in A_0$.

1.1. **Graded Hurwitz series.** A *Hurwitz series* over A is a formal power series in the form

$$\varphi(u) = \sum_{k\geqslant 0} \varphi_k \frac{u^k}{k!} \ \in \ A \otimes \mathbb{Q}[[u]]$$

with $\varphi_k \in A$ for all $k = 0, 1, 2, 3, \ldots$ (see [28]). Set $\deg u = 2$. The series $\varphi(u)$ has the degree $2l$ if $\deg \varphi_k + 2k = 2l$ for any k. Let us denote the set of Hurwitz series over A by $HA[[u]]$ and the set of graded Hurwitz series over A by $H^{gr}A[[u]] = \sum H^{gr}_{2l}A[[u]]$.

Properties of Hurwitz series:

1) Hurwitz series over A form a commutative associative ring $HA[[u]]$ with respect to the usual addition and multiplication of series.

2) The units of the ring $HA[[u]]$ are the elements $\psi(u)$ such that $\psi(0)$ is a unit of A.

3) This ring is closed under differentiation and integration with respect to u.

4) If $\varphi(u) \in HA[[u]]$, $\varphi(0) = 0$, $\psi(u) \in HB[[u]]$, then $\psi(\varphi(u)) \in H(A \otimes B)[[u]]$.

If $\varphi(u) \in HA[[u]]$, $\varphi(0) = 0$, $\varphi'(0)$ is a unit of A, then $\varphi^{-1}(u) \in HA[[u]]$, where $\varphi^{-1}(u)$ is defined by $\varphi(\varphi^{-1}(u)) = u$.

1.2. **The formal group law.** A commutative one-dimensional *formal group law* over the ring A is a formal series $F(t_1, t_2) = t_1 + t_2 + \sum_{i,j} \alpha_{i,j} t_1^i t_2^j$, $\alpha_{i,j} \in A$, such that the following conditions hold:

$$F(t, 0) = t, \quad F(t_1, t_2) = F(t_2, t_1), \quad F(t_1, F(t_2, t_3)) = F(F(t_1, t_2), t_3).$$

For brevity we will use "formal group" as synonim to "formal group law".

A *homomorphism* of formal groups $h : F_1 \to F_2$ over a ring A is a formal series $h(u) \in A[[u]]$ where $h(0) = 0$ such that $h(F_1(u, v)) = F_2(h(u), h(v))$. The homomorphism h is an *isomorphism* if $h'(0)$ is a unit in A and a *strong isomorphism* if $h'(0) = 1$.

For each formal group $F \in A[[t_1, t_2]]$ there exists a strong isomorphism $f : L \to F$ over the ring $A \otimes \mathbb{Q}$, where L is the linear group $L(u, v) = u + v$. It is uniquely defined by the condition

$$f(u + v) = F(f(u), f(v)). \tag{1}$$

The function $f(u)$ is called the *exponential* of the formal group F. The function $g(t) = f^{-1}(t)$ is called the *logarithm* of the formal group F. From (1) we have

$$f'(u) = \frac{\partial}{\partial t_2} F(f(u), t_2)|_{t_2=0}\,, \qquad \frac{1}{g'(t)} = \frac{\partial}{\partial t_2} F(t, t_2)|_{t_2=0}\,. \tag{2}$$

Thus, $g'(t)$ and $f'(g(t)) \in A[[t]]$. Therefore $g(t)$ and $f(u)$ are Hurwitz series. If F is graded of degree 2, (that is $\deg \alpha_{i,j} = -2(i+j-1)$), then $f(u) \in H_2^{gr} A[[u]]$ and $g(t) \in H_2^{gr} A[[t]]$.

Let F_1 and F_2 be formal groups over A. There exists a strong isomorphism $h : F_1 \to F_2$ over A if and only if $f_2\big(f_1^{-1}(t)\big) \in A[[t]]$. Notice that if $h : F_1 \to F_2$ a strong isomorphism, and f_1 is the exponential of F_1, then $h(f_1(u + v)) = F_2(h(f_1(u)), h(f_1(v)))$, that is $h \circ f_1$ is the exponential of F_2. Thus there exists a strong isomorphism $h : L \to F_2$ over A if and only if the exponential $f(u)$ is a series over A.

Let $f(u) \in H_2^{gr} A[[u]]$, $f(0) = 0$, $f'(0) = 1$, and $F(t_1, t_2) = f(f^{-1}(t_1) + f^{-1}(t_2))$. We have $F(t_1, t_2) = t_1 + t_2 + \sum_{n\geq 1} \sum_{i+j=n+1} \alpha_{i,j} t_1^i t_2^j \in H_2^{gr} A[[t_1, t_2]]$. The coefficients $\alpha_{i,j}$ are polynomials of f_i, where $f(u) = u + \sum_{i\geq 1} f_i u^{i+1}$. By the construction $\alpha_{i,j} \in A_{-2(i+j-1)} \otimes \mathbb{Q}$. Denote by A_f the subring in $A \otimes \mathbb{Q}$ generated by $\alpha_{i,j}$ and by B_f the subring in $A \otimes \mathbb{Q}$ generated by f_i. The ring B_f is by the construction the smallest ring over which there exists a strong isomorphism between L and F. Thus we have $A_f \subset B_f \subset A \otimes \mathbb{Q}$.

Thus, *the problem of description of graded formal groups over a torsion free ring A is equivalent to the problem of description of such $f(u) \in H_2^{gr} A[[u]]$, that $F(t_1, t_2) \in A[[t_1, t_2]]$*. Here arises the *problem of description of the graded rings A_f and B_f.*

1.3. **The universal formal group law.**

A formal group law $\mathcal{F}(t_1,t_2) = t_1 + t_2 + \sum a_{i,j}t_1^i t_2^j$ over the ring $\mathcal{A}$ is the *universal formal group law* if for any formal group $F(t_1,t_2)$ over any ring A there exists an unique ring homomorphism $r : \mathcal{A} \to A$ such that $F(t_1,t_2) = t_1 + t_2 + \sum r(a_{i,j})t_1^i t_2^j$.

Thus, the *problem of description of formal groups over A can be brought to the problem of description of ring homomorphisms $\mathcal{A} \to A$.*

Consider the graded ring $\mathcal{U} = \mathbb{Z}[\beta_{i,j}],\ i > 0,\ j > 0,\ \deg \beta_{i,j} = -2(i+j-1)$ and the series

$$\Phi(t_1,t_2) = t_1 + t_2 + \sum \beta_{i,j}t_1^i t_2^j.$$

Set $\Phi^l(t_1,t_2,t_3) = \Phi\big(\Phi(t_1,t_2),t_3\big)$ and $\Phi^r(t_1,t_2,t_3) = \Phi\big(x,\Phi(y,z)\big)$. We obtain

$$\Phi^l(t_1,t_2,t_3) = t_1 + t_2 + t_3 + \sum \beta^l_{i,j,k}t_1^i t_2^j t_3^k,$$
$$\Phi^r(t_1,t_2,t_3) = t_1 + t_2 + t_3 + \sum \beta^r_{i,j,k}t_1^i t_2^j t_3^k$$

where $\beta^l_{i,j,k}$ and $\beta^r_{i,j,k}$ are homogeneous polynomials of $\beta_{i,j}$, $\deg \beta^l_{i,j,k} = \deg \beta^r_{i,j,k} = -2(i+j+k-1)$.

Let $J \subset \mathcal{U}$ be the ideal of associativity generated by polynomials $\beta^l_{i,j,k} - \beta^r_{i,j,k}$. Consider the ring $\mathcal{A} = \mathcal{U}/J$ and the canonical projection $\pi\colon \mathcal{U} \to \mathcal{A}$. Denote by $\mathcal{F}(t_1,t_2)$ the series $\pi[\Phi](t_1,t_2) = t_1 + t_2 + \sum \pi(\beta_{i,j})t_1^i t_2^j$.

By the construction, the formal group $\mathcal{F}(t_1,t_2)$ over the ring $\mathcal{A}$ is the universal formal group. Therefore, by Lazard's theorem (see [21]) we obtain that

$$\mathcal{A} = \mathbb{Z}[a_n],\ n = 1,2,\ldots,\ \deg a_n = -2n.$$

Thus, the *problem of description of formal groups over the ring A is equivalent to the problem of description of ring homomorphisms $\mathbb{Z}[a_n] \to A$.*

Consider the exponential $f_{\mathcal{U}}(u) = u + \sum b_n u^{n+1} \in \mathcal{A} \otimes \mathbb{Q}[[u]]$ of the group $\mathcal{F}(t_1,t_2)$ and the ring $\mathcal{B} = \mathbb{Z}[b_n] \subset \mathcal{A} \otimes \mathbb{Q}$.

Thus the problem of description of formal groups over torsion-free rings in terms of their exponentials can be presented in the following way:

For a formal group $F(t_1,t_2) = \sum \alpha_{i,j}t_1^i t_2^j$ over the torsion-free ring A the classifying group homomorphism holds:

$$\phi : \mathcal{A} \longrightarrow A.$$

It is defined by the condition $\phi(\beta_{i,j}) = \alpha_{i,j}$.

Having $F(t_1,t_2) \in A[[t_1,t_2]]$, we find $g(t) \subset A \otimes \mathbb{Q}[[t]]$ from (2) and the exponential $g^{-1}(u) = f(u) = u + \sum f_n u^{n+1}$. Because $f_{\mathcal{U}}(u) = u + \sum b_n u^{n+1}$ we get $\phi(b_n) = f_n$.

We obtain the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_F & \longrightarrow & \mathcal{A} & \longrightarrow & A_f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widehat{J}_F & \longrightarrow & \mathcal{B} & \longrightarrow & B_f & \longrightarrow & 0
\end{array}$$

The problems of the description of the kernels J_F, $\widehat{J}_F$ and the images A_f, B_f for $\mathcal{A}$, $\mathcal{B}$ arise.

The classical Example 1. Let $f(u) = \frac{1}{\mu}(1 - e^{-\mu u}) \in H_2^{gr}A[[u]]$ where $A = \mathbb{Z}[\mu]$, $\deg \mu = -2$. Then we have $g(t) = -\frac{1}{\mu}\ln(1-\mu t) \in H_2^{gr}A[[t]]$ and $F(t_1, t_2) = t_1 + t_2 - \mu t_1 t_2$ is a formal group over $\mathbb{Z}[\mu]$. Then $A_f = A$ and B_f is isomorhic to $\mathbb{Z}[b_1, \ldots, b_n, \ldots]/J$ such that $A_f \to B_f : \mu \mapsto -2b_1$. We have $f_i = (-1)^i \frac{\mu^i}{(i+1)!}$, thus J is generated by polynomials $(i+1)!b_i - 2^i b_1^i$, $i = 2, 3, \ldots$

Denote by $\mathcal{A}^{(1)} = \sum \mathcal{A}^{(1)}_{-2n}$ the quotient ring of the ring $\mathcal{A}$ modulo an ideal I, which is generated by the elements, that can be represented as a product of at least two elements of positive degree. Let $\widetilde{\pi} : \mathcal{A} \to \mathcal{A}^{(1)}$ be the canonical projection. We obtain
$\widetilde{\pi}[f_{\mathcal{U}}^{-1}](t) = t - \sum b_n t^{n+1}$ and

$$\widetilde{\pi}[\mathcal{F}](t_1, t_2) = t_1 + t_2 + \sum b_n((t_1 + t_2)^{n+1} - t_1^{n+1} - t_2^{n+1}).$$

Thus the subgroup $\widetilde{\pi}(\mathcal{A}_{-2n})$ of the group $\mathcal{A}^{(1)}_{-2n}$ is isomorphic to the group $\mathbb{Z}$ with generators $\nu(n+1)\widetilde{\pi}(b_n)$, where $\nu(n) = \gcd \begin{pmatrix} n \\ k \end{pmatrix}, k = 1, 2, \ldots, n-1$.

There exist multiplicative generators a_n^*, $n = 1, 2, \ldots$ of the ring $\mathcal{A} = \mathbb{Z}[a_n]$, such that the embedding $\mathcal{A} \to \mathcal{B}$ is given by the formula $a_n^* = \nu(n+1)b_n^*$, where b_n^*, $n = 1, 2, \ldots$ are the multiplicative generators of the ring $\mathcal{B} = \mathbb{Z}[b_n]$.

The choice of generators in $\mathcal{A}$ (see survey [9]).

For any formal group $F(t_1, t_2)$ over A one can define *the power system* (see [8]):
$[t]_n \in A[[t]]$ for $n \in \mathbb{Z}$, where $[t]_n = nt + (t^2)$, $[t]_0 = 0$, $[t]_1 = t$, and $[t]_n$ are defined recursively by the condition $[t]_n = F(t, [t]_{n-1})$, $n = 2, 3, 4, \ldots$, and $n = 0, -1, -2, \ldots$.
We will denote $\bar{t} = [t]_{-1}$. Over the torsion-free ring A we have $[t]_k = f(k\, g(t))$.

Let p be a prime number, $\mathbb{Z}_{(p)}$ - the ring of integer p-adic numbers. In the polynomial ring $\mathcal{A}_{(p)} = \mathcal{A} \otimes \mathbb{Z}_{(p)}$ the multiplicative generators can be chosen in the following way: Set

$$\frac{\partial}{\partial t_2}\mathcal{F}(t, t_2)|_{t_2=0} = \sum_n c_n t^n, \qquad [t]_p = \sum_n \widehat{c}_n t^{n+1}, \tag{3}$$

then $\mathcal{A}_{(p)} = \mathbb{Z}_{(p)}[\widehat{a}_n]$, where $\widehat{a}_n = c_n$ for $n \neq p^q - 1$ and $\widehat{a}_n = \widehat{c}_n$ for $n = p^q - 1$, $q = 1, 2, 3, \ldots$.

For $p = 2$ the generators can be also chosen in the following way:
Let $\gamma(t) = \bar{t}$. We have $\gamma(t) = -t + \sum_{i \geq 1} \gamma_i t^{i+1}$. Then $\mathcal{A}_{(2)} = \mathbb{Z}_{(2)}[\widehat{a}_n]$, where $\widehat{a}_n = c_n$ for $n \neq 2^q - 1$ and $\widehat{a}_n = \gamma_{2^q-1}$ for $n = 2^q - 1$, $q = 1, 2, \ldots$.

The multiplicative generators of $\mathcal{A}_{(p)}$ in the dimentions $-2(p^q - 1)$ can be chosen using the coefficients of the series $[t]_{1-p}$ instead of the coefficients of the series $[t]_p$. In the case $p = 2$ these are the coefficients of $\bar{t}$. The proof follows from the identity $F([t]_p, [t]_{1-p}) = t$ and the fact that $[t]_p = pt +$ (series with coefficients of negative degree).

Lemma 2. *Let $f(u) \in HA[[u]]$, $f(0) = 0$, $f'(0) = 1$. Then $f(g(t_1) + g(t_2)) \in A_{(p)}$.*

Proof. Consider the formal group $F(t_1, t_2) = f(g(t_1) + g(t_2)) \in A \otimes \mathbb{Q}[[t_1, t_2]]$ and the homomorphism $\mathcal{A} \to A \otimes \mathbb{Q}$, classifying this group. Using the choice of multiplicative generators of the ring $\mathcal{A}$ described above, we get that the image of the ring $\mathcal{A}$ lies in $A_{(p)}$.

1.4. **The formal group of geometric cobordisms.** Consider the ring of complex cobordisms Ω_U of stably-complex manifolds (see [32], [26]). According to Milnor-Novikov theorem we have $\Omega_U \simeq \mathbb{Z}[a_n],\ n = 1, \dots,\ \deg a_n = -2n$.

Let $\eta \to \mathbb{C}P^n,\ n \leqslant \infty$ be the canonical complex line bundle over an n-dimentional complex projective space.

In the theory of complex cobordisms the isomorphism $U^*(\mathbb{C}P^n) = \Omega_U[t]/(t^{n+1})$ takes place, where $t = c_1(\eta)$ is the first Chern class. The formal group

$$\mathcal{F}_U(t_1, t_2) = t_1 + t_2 + \sum c_{i,j} t_1^i t_2^j, \quad \deg c_{i,j} = -2(i + j - 1), \tag{4}$$

of complex cobordisms is given by series (see [25])

$$c_1(\eta_1 \otimes \eta_2) = \mathcal{F}_U(t_1, t_2) \in U^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \Omega_U[[t_1, t_2]].$$

The logarithm of this group (the series of A.S. Mishchenko) takes the form

$$g(t) = \sum_{n \geqslant 0} [\mathbb{C}P^n] \frac{t^{n+1}}{n+1}. \tag{5}$$

In the work [25] the Adams-Novikov operators Ψ_U^k in the theory of complex cobordisms were introduced in therms of formal groups $\mathcal{F}_U(t_1, t_2)$. This operators are defined by the formulas $\Psi_U^k t = \frac{1}{k}[t]_k,\ k \neq 0$, and $\Psi_U^0 t = g(t)$.
Notice that $\Psi_U^k t \in U^*(\mathbb{C}P^\infty) \otimes \mathbb{Z}[\frac{1}{k}],\ k \neq 0$, and $\Psi_U^0 t \in U^*(\mathbb{C}P^\infty) \otimes \mathbb{Q}$.

S.P. Novikov showed the important role of the operators Ψ_U^k for the Adams-Novikov spectral sequence and for the description of cobordism classes of manifolds with a group action in terms of fixed points of this action.

D. Quillen published in [29] the fundamental observation, that the homomorphism $\varphi\colon \mathcal{A} \to \Omega_U$, classifying the formal group $\mathcal{F}_U(t_1, t_2)$, is an isomorphism, thus the formal group of complex cobordisms was identified with the universal formal group. In this work on the basis of the results of the algebraic theory of formal groups an effective description of the important theory of Brown-Peterson cohomology was obtained. The applications of the theory of formal groups formed the background of a powerful direction in algebraic topology (for the foundations of this direction see the survey [9]).

1.5. **Hirzebruch genera and formal groups.** Let

$$f(u) = u + \sum_{k\geq 1} f_k u^{k+1}, \quad \text{where} \quad f_k \in A \otimes \mathbb{Q}.$$

The formal series

$$\prod_{i=1}^{n} \frac{u_i}{f(u_i)}$$

is invariant with respect to the permutation of the variables $u_1, \dots, u_n$, and therefore it can be presented in the form $L_f(\sigma_1, \dots, \sigma_n)$, where σ_k is the k-th elementary symmetric polynomial of $u_1, \dots, u_n$.

The Hirzebruch genus $L_f(M^{2n})$ *of a stably complex manifold* M^{2n} is the value of the cohomology class $L_f(c_1, \dots, c_n)$ on the fundamental cycle $\langle M^{2n} \rangle$ of the manifold M^{2n}, where c_k is the k-th Chern class of the tangent bundle of the manifold M^{2n}. The foundations of the theory and the applications of the Hirzebruch genera were laid in the work [13].

Each Hirzebruch genus L_f defines a ring homomorphism

$$L_f \colon \Omega_U \to A \otimes \mathbb{Q}$$

and for any ring homomorphism $\varphi \colon \Omega_U \to A \otimes \mathbb{Q}$ there exists a series $f(u) \in A \otimes \mathbb{Q}[[u]]$, $f(0) = 0,\ f'(0) = 1$ such that $\varphi = L_f$.

The Hirzebruch genus L_f is called A-*integer* if $L_f(M^{2n}) \in A_{-2n}$ for any stably complex manifold M^{2n}.

Identifying the formal group of complex cobordisms with the universal formal group, we obtain (see details in [8]):

The correspondence of the formal group $F(t_1, t_2)$ over the ring A to its exponential $f(u) \in HA[[u]]$ gives a one-one mapping of the formal groups $F(t_1, t_2)$ over the ring A and the Hirzebruch genera L_f, taking values in A.

Thus, *the problem of description of series* $f(u) \in HA[[u]]$ *such that* $F(t_1, t_2) \in A[[t_1, t_2]]$ (see section 1.2), *is equivalent to the problem of description of* A*-integer Hirzebruch genera.*

The importance and the celebrity of this problem (the famous Atiyah–Singer theorem and its developments) is connected with the Hirzebruch genera, the values of which on manifolds is equal to indices of fundamental differential operators on this manifolds.

S.P. Novikov in [24] showed that for numerical Hirzebruch genera the formula holds:

$$\sum L_f([\mathbb{C}P^n]) \frac{t^{n+1}}{n+1} = g(t).$$

Thus if we know the formal group we obtain the values of $L_f([\mathbb{C}P^n])$ using the formula (2). In the general case it is shown in [10] that the identity map $\Omega_U \to \Omega_U$ gives the Chern-Dold character

$$ch_U : U^*(X) \to H^*(X; \Omega_U(\mathbb{Z})) \subset H^*(X; \Omega_U \otimes \mathbb{Q}),$$

where $H^*(\cdot\,; \Omega_U \otimes \mathbb{Q})$ is the classical cohomology theory with coefficients in $\Omega_U \otimes \mathbb{Q}$ and $\Omega_U(\mathbb{Z}) \subset \Omega_U \otimes \mathbb{Q}$ is the ring generated by the elements of $\Omega_U \otimes \mathbb{Q}$ with integer Chern numbers. In [10] it is shown that $\Omega_U(\mathbb{Z})$ is the ring of polynomials generated by $\frac{[\mathbb{C}P^n]}{n+1}$, $n = 1, 2, \ldots$ and the Chern-Dold character is defined by the formula $ch_U t = f(u)$, where $f(u)$ is the exponential of the universal formal group law.

Example 3. The series

$$f(u) = \frac{e^{\alpha u} - e^{\beta u}}{\alpha e^{\alpha u} - \beta e^{\beta u}}$$

is the exponential of the formal group

$$F(t_1, t_2) = \frac{t_1 + t_2 - a t_1 t_2}{1 - b t_1 t_2}, \quad \text{where } a = \alpha + \beta,\ b = \alpha\beta,$$

corresponding to the remarkable two-parameter Todd genus $T_{\alpha,\beta}\colon \Omega_U \to \mathbb{Z}[a, b]$, a particular case of which are the famous Hirzebruch genera: the Todd genus ($b = 0$); the signature ($a = 0$); the Eulerian characteristic ($a = 2c,\ b = c^2$). See Example (63) below.

Let $f_k(u) \in A \otimes Q[[u]],\ k = 1, 2$, be the exponentials of formal group laws $F_1(t_1, t_2)$ and $F_2(t_1, t_2)$ over A. The series $f_1(u)$ and $f_2(u)$ define Hirzebruch genera L_1 and L_2, *equivalent* over A, if $f_2(u) = \psi(u) f_1(u)$, where $\psi(u) \in A[[u]]$.

Examples of series $f_1(u)$ and $f_2(u)$ defining equivalent Hirzebruch genera over A such that the formal groups $F_1(t_1, t_2)$ and $F_2(t_1, t_2)$ are not strongly isomorphic can be easily found. The exponentials $f_1(u)$ and $f_2(u)$ of formal group laws $F_1(t_1, t_2)$ and $F_2(t_1, t_2)$ define *strongly equivalent* over A Hirzebruch genera L_1 and L_2 if

$$f_2(u) = \psi(u) f_1(u) \quad \text{and} \quad f_2(u) = h\big(f_1(u)\big),$$

where $\psi(u) \in A[[u]]$, $h(t) \in A[[t]]$, that is L_1 and L_2 are equivalent over A, and the formal group laws $F_1(t_1, t_2)$ and $F_2(t_1, t_2)$ are strongly isomorphic over A.

Example 4. The classical $\mathcal{A}$ and $\mathcal{L}$ Hirzebruch genera correspond to the series

$$f_1(u) = 2 \sinh \frac{u}{2} \quad \textit{and} \quad f_2(u) = \tanh u,$$

which define formal group laws over $\mathbb{Z}[\frac{1}{2}]$:

$$F_1(t_1, t_2) = t_1 \sqrt{1 - \frac{1}{4} t_2^2} + t_2 \sqrt{1 - \frac{1}{4} t_1^2}\,,$$
$$F_2(t_1, t_2) = \frac{t_1 + t_2}{1 - t_1 t_2}.$$

It is well known that the Hirzebruch genera $\mathcal{A}$ and $\mathcal{L}$ are strongly equivalent over $\mathbb{Z}[\frac{1}{2}]$.

2. **The elliptic formal group law.**

In this section we will solve some problems considered above in the particular case of a family of formal groups over $E = \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$ defined by the elliptic curve. Here $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_6)$ are the parameters of the elliptic curve.

2.1. **The elliptic curve.** Consider the elliptic curve $\mathcal{V}$, given in Weierstrass parametrization by the equation

$$y^2 + \mu_1 xy + \mu_3 y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6. \tag{6}$$

It is a plane algebraic curve in variables x, y, homogeneous in respect to the degrees $\deg x = -4$, $\deg y = -6$, $\deg \mu_i = -2i$. Its compactification in $\mathbb{C}P^2$ is given in homogeneous coordinates $(X : Y : Z)$ by the equation

$$Y^2 Z + \mu_1 XYZ + \mu_3 YZ^2 = X^3 + \mu_2 X^2 Z + \mu_4 XZ^2 + \mu_6 Z^3. \tag{7}$$

The corresponding degrees are $\deg X = -4$, $\deg Y = -6$, $\deg Z = 0$, $\deg \mu_i = -2i$.

Note. There exists also a remarkable parametrisation of an elliptic curve in the Hessian form (see [31]):

$$X^3 + Y^3 + Z^3 = dXYZ.$$

Let us fix the Weierstrass form (7) of the equation and now on consider μ_i as algebraically independent variables, unless otherwise stipulated.

In $\mathbb{C}P^2$ there exists three canonical coordinate maps.

In the coordinate map $Z \neq 0$ with the coordinates $x = X/Z$ and $y = Y/Z$ the equation of the curve (7) takes the form (6). In this map the curve can be uniformized by the Weierstrass functions (see section 2.6).

In the coordinate map $Y \neq 0$ with the coordinates $t = -X/Y$ and $s = -Z/Y$ the equation of the curve (7) takes the form

$$s = t^3 + \mu_1 ts + \mu_2 t^2 s + \mu_3 s^2 + \mu_4 ts^2 + \mu_6 s^3. \tag{8}$$

Here t and s are the arithmetic Tate coordinates (see [33]). Thus t is the uniformizing coordinate for the curve (8). The remarkable fact about the coordinates $(t, s(t))$ is that the elliptic formal group law in this coordinates is given by series over E. We have $\deg t = 2$, $\deg s = 6$.

In the coordinate map $X \neq 0$ with the coordinates $v = Y/X$ and $w = Z/X$ the equation of the curve (7) takes the form

$$vw(v + \mu_1 + \mu_3 w) = 1 + \mu_2 w + \mu_4 w^2 + \mu_6 w^3. \tag{9}$$

We have $\deg v = -2$, $\deg w = 4$.

The discriminant of the elliptic curve (6) is Δ, where

$$\begin{aligned} 4\Delta = (\mu_1\mu_3 + 2\mu_4)^2(4\mu_2 + \mu_1^2)^2 - 32(\mu_1\mu_3 + 2\mu_4)^3 - 108(4\mu_6 + \mu_3^2)^2 + \\ + 36(\mu_1\mu_3 + 2\mu_4)(4\mu_2 + \mu_1^2)(4\mu_6 + \mu_3^2) - (4\mu_2 + \mu_1^2)^3(4\mu_6 + \mu_3^2). \end{aligned} \tag{10}$$

Let

$$\mu_2 = -(e_3 + e_2 + e_1) - \frac{1}{4}\mu_1^2, \quad \mu_4 = e_3e_2 + e_3e_1 + e_2e_1 - \frac{1}{2}\mu_1\mu_3, \quad \mu_6 = -e_1e_2e_3 - \frac{1}{4}\mu_3^2.$$

In this notation the equation (6) becomes

$$(2y + \mu_1 x + \mu_3)^2 = 4(x - e_1)\,(x - e_2)\,(x - e_3).$$

We get

$$\Delta = 16\,(e_1 - e_2)^2\,(e_1 - e_3)^2\,(e_2 - e_3)^2.$$

2.2. **The geometric group structure on an elliptic curve.** (See [34]).

For the geometric group structure on the elliptic curve we have $P_1 + P_2 + P_3 = 0$ for three points P_1, P_2, P_3 when the points P_1, P_2 and P_3 lie on a straight line. Let the point O with coordinates $(0 : 1 : 0)$ on the elliptic curve be the zero of the group. Consequently we have $P_1 + P_2 = \overline{P_3}$ in the group when $P_1 + P_2 + P_3 = 0$ and $P_3 + \overline{P_3} + O = 0$.

The classical geometric group structure on the elliptic curve in Tate coordinates gives a remarkable formal group law $F_\mu(t_1, t_2)$ over E that will be called *the general elliptic formal group law.* The corresponding formal group law induced by conditions on μ_1, μ_2, μ_3, μ_4, μ_6 will be called *the elliptic formal group law.*

In the coordinate map $Y \neq 0$ let (t_1, s_1), (t_2, s_2), (t_3, s_3) and $(\overline{t_3}, \overline{s_3})$ be the coordinates of the points P_1, P_2, P_3 and $\overline{P_3}$ respectively. The general elliptic formal group law F_μ is defined by the condition $F_\mu(t_1, t_2) = \overline{t_3}$. In this coordinates $F_\mu(t_1, t_2)$ is a series depending on t_1, t_2 (see [33]). Due to the construction of the addition law on the elliptic curve, the series $F_\mu(t_1, t_2)$ determines a commutative one-dimentional formal group. Let $F_\mu(t_1, t_2) = t_1 + t_2 + \sum_{i,j} \alpha_{i,j} t_1^i t_2^j$. It is known that $\alpha_{i,j} \in E$.

One can find the explicit multiplication law for the Hessian form in [31].

2.3. **The general elliptic formal group law.** In the coordinate map $Y \neq 0$ the curve $\mathcal{V}$ is given by the equation (8):

$$s = t^3 + \mu_1 ts + \mu_2 t^2 s + \mu_3 s^2 + \mu_4 ts^2 + \mu_6 s^3.$$

The function $s(t)$ is defined by (8) and the condition $s(0) = 0$. The series expansion at $t = 0$ will be

$$s = t^3 + \mu_1 t^4 + (\mu_1^2 + \mu_2)t^5 + (\mu_3 + 2\mu_2\mu_1 + \mu_1^3)t^6 + (t^7) \tag{11}$$

Let $s = mt + b$ be the equation of the straight line that contains the points P_1, P_2 and P_3 with coordinates (t_1, s_1), (t_2, s_2) and (t_3, s_3) respectively. Using the points P_1 and P_2 we obtain

$$m = \frac{s_1 - s_2}{t_1 - t_2}, \qquad b = \frac{t_1 s_2 - t_2 s_1}{t_1 - t_2}. \tag{12}$$

Using the equation (8) we get a cubic equation on t

$$\xi_0(m)t^3 + \xi_2(m, b)t^2 + \xi_4(m, b)t + \xi_6(b) = 0 \tag{13}$$

with the roots t_1, t_2 and t_3. Here

$$\xi_0(m) = 1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3, \qquad \xi_2(m,b) = \mu_1 m + \mu_2 b + \mu_3 m^2 + 2\mu_4 mb + 3\mu_6 m^2 b,$$
$$\xi_6(b) = -b(1 - \mu_3 b - \mu_6 b^2), \qquad \xi_4(m,b) = -m + \mu_1 b + 2\mu_3 mb + \mu_4 b^2 + 3\mu_6 mb^2.$$

Let

$$\eta_2(m) = \mu_2 + \mu_4 m + \mu_6 m^2, \quad \eta_1(m,b) = \mu_1 + \mu_3 m + \mu_4 b + 2\mu_6 mb, \quad \eta_0(b) = (1 - \mu_3 b - \mu_6 b^2).$$

Thus

$$\xi_0(m) = 1 + m\eta_2(m), \qquad \xi_4(m,b) = -m\eta_0(b) + b\eta_1(m,b),$$
$$\xi_2(m,b) = m\eta_1(m,b) + b\eta_2(m), \qquad \xi_6(b) = -b\eta_0(b).$$

We get the relations on the coefficients of (13)

$$\xi_6(b) - t_1 t_2 \xi_2(m,b) = t_1 t_2 (t_1 + t_2)\xi_0(m), \tag{14}$$

$$t_1^2 t_2^2 \xi_0(m) = (t_1 + t_2)\xi_6(b) + t_1 t_2 \xi_4(m,b). \tag{15}$$

We have $\xi_0(m)(t_1 + t_2 + t_3) = -\xi_2(m,b)$, thus

$$t_3 = -\frac{(t_1 + t_2)\xi_0(m) + \xi_2(m,b)}{\xi_0(m)}. \tag{16}$$

Let $s = nt$ be the equation of the straight line that contains the points P_3, $\overline{P_3}$ and O with coordinates (t_3, s_3), $(\overline{t_3}, \overline{s_3})$ and $(0,0)$ respectively. Using the points P_3 and O we obtain $n = s_3/t_3 = m + \frac{b}{t_3}$. Using the equation (8) we get a quadratic equation on t

$$\xi_0(n)t^2 + \xi_2(n,0)t + \xi_4(n,0) = 0.$$

with the roots t_3 and $\overline{t_3}$.

Thus

$$\overline{t_3} = -\frac{n}{t_3 \xi_0(n)}.$$

From (12) we see that b, m can be represented as series of t_1 and t_2 with $\deg b = 6$ and $\deg m = 4$. From (16) we see that t_3 can be represented as series of t_1 and t_2 with $\deg t_3 = 2$. Let us remind $\deg t_i = 2$.

From (16) and (14) have $n = m + t_1 t_2 \frac{(1+\mu_2 m + \mu_4 m^2 + \mu_6 m^3)}{(1 - \mu_3 b - \mu_6 b^2)}$, thus n can be represented as series of t_1 and t_2 with $\deg n = 4$.

From this formulas (using (15)) we obtain

$$F_\mu(t_1, t_2) = ((t_1 + t_2)\eta_0(b) - t_1 t_2 \eta_1(m,b))\,\frac{\xi_0(m)}{\xi_0(n)\eta_0(b)^2}.$$

Thus we obtain:

Theorem 5. *The general elliptic formal group law $F_\mu(t_1, t_2)$ is given by the formula*

$$F_\mu(t_1,t_2) = (t_1+t_2-\mu_1 t_1 t_2-\mu_3((t_1+t_2)b+t_1t_2m)-\mu_4t_1t_2b-\mu_6b((t_1+t_2)b+2t_1t_2m))\times \times\frac{(1+\mu_2 m+\mu_4 m^2+\mu_6 m^3)}{(1+\mu_2 n+\mu_4 n^2+\mu_6 n^3)(1-\mu_3 b-\mu_6 b^2)^2}. \quad (17)$$

Corollary 6. *Let $F_\mu(t_1,t_2) = t_1 + t_2 + \sum_{i,j}\alpha_{i,j}t_1^i t_2^j$ be the general elliptic formal group law. Then $\alpha_{i,j} \in E$.*

Let us denote $p = t_1 t_2$. Using the formulas (14) and (15), we obtain from (17)

$$F_\mu(t_1,t_2) = \frac{((t_1+t_2)\eta_0(b) - p(\mu_1+\mu_3 m+\mu_4 b+2\mu_6 bm))}{\eta_0(b)(\eta_0(b)+p(\mu_2+2\mu_4 m+3\mu_6 m^2)+(\mu_4+3\mu_6 m)p^2\frac{\xi_0(m)}{\eta_0(b)}+\mu_6 p^3\frac{\xi_0(m)^2}{\eta_0(b)^2}))}. \quad (18)$$

Example 7. In the case $(\mu_4, \mu_6) = (0, 0)$ we get

$$F_\mu(t_1,t_2) = \frac{(t_1+t_2)(1-\mu_3 b)-\mu_1 t_1 t_2-\mu_3 t_1 t_2 m}{(1-\mu_3 b)(1+\mu_2 t_1 t_2-\mu_3 b)}. \quad (19)$$

Using the formulas (14) and (15), we obtain from (17)

$$F_\mu(t_1,t_2) = \frac{(t_1+t_2)(1+\mu_2 m+\mu_4 m^2+\mu_6 m^3)+m(\mu_1+\mu_3 m)+b(\mu_2+2\mu_4 m+3\mu_6 m^2)}{(1+\mu_2 m+\mu_4 m^2+\mu_6 m^3)(1-\mu_3 b)-\frac{b}{p}(\mu_1+\mu_3 m)(1-\mu_3 b-\mu_6 b^2)}. \quad (20)$$

Example 8. In the case $(\mu_1, \mu_3) = (0, 0)$ we get

$$F_\mu(t_1,t_2) = t_1+t_2+b\frac{(\mu_2+2\mu_4 m+3\mu_6 m^2)}{(1+\mu_2 m+\mu_4 m^2+\mu_6 m^3)}. \quad (21)$$

We will analyze the general elliptic formal group law (17) and the formal group laws (19), (21).

2.4. **The equations on the exponentials of the elliptic formal group laws.**

Using (11) and (17), we get

$$\frac{\partial}{\partial t_2}F_\mu(t,t_2)\Big|_{t_2=0} = 1-\mu_1 t-\mu_2 t^2-2\mu_3 s-2\mu_4 ts-3\mu_6 s^2. \quad (22)$$

Let $u = g(t)$ and $\phi(u) = s(f(u))$, where $s(t)$ is defined by (8) as before. By (2) we come to the system

$$\begin{cases} f'(u) = 1-\mu_1 f(u)-\mu_2 f(u)^2-2\mu_3\phi(u)-2\mu_4 f(u)\phi(u)-3\mu_6\phi(u)^2 \\ \phi(u) = f(u)^3+\mu_1 f(u)\phi(u)+\mu_2 f(u)^2\phi(u)+\mu_3\phi(u)^2+\mu_4 f(u)\phi(u)^2+\mu_6\phi(u)^3. \end{cases} \quad (23)$$

Let $M(t) = 1-\mu_1 t-\mu_2 t^2$ and $N(t) = \mu_3+\mu_4 t$. We obtain the following result:

Theorem 9. *Let $(\mu_3, \mu_4, \mu_6) \neq (0, 0, 0)$. Then the exponential $f(u)$ of the elliptic formal group law F_μ is the solution of the equation*

$$\mu_6 \left[f'^3 + 3M(f)f'^2 - 4M(f)^3 + 18M(f)N(f)f^3 + 27\mu_6 f^6\right] = -N(f)^2 \left[f'^2 - M(f)^2 + 4N(f)f^3\right], \tag{24}$$

with the initial condition $f(0) = 0$ and the condition $f'(0) = 1$ which fixes the branch of solutions.

Let $(\mu_3, \mu_4, \mu_6) = (0, 0, 0)$. Then the exponential $f(u)$ of the elliptic formal group law F_μ is the solution of the equation

$$f' = M(f) \tag{25}$$

with the initial condition $f(0) = 0$.

Example 10. Let $\mu_6 = 0$, $(\mu_3, \mu_4) \neq (0, 0)$. Then the equation (24) gives

$$f'^2 = 1 - 2\mu_1 f + (\mu_1^2 - 2\mu_2)f^2 + (2\mu_1\mu_2 - 4\mu_3)f^3 + (\mu_2^2 - 4\mu_4)f^4. \tag{26}$$

Remark that for $\mu_3 = \mu_4 = 0$ this equation leads to the equation (25) (the sign is determined by the conditions $f(0) = 0$, $f'(0) = 1$), so the equation (26) should be considered as the general equation on the exponential in the case $\mu_6 = 0$.

Example 11. Let $(\mu_1, \mu_2, \mu_3) = (0, 0, 0)$. Then the equation (24) gives

$$\mu_6 f'^3 + (\mu_4^2 f^2 + 3\mu_6)f'^2 + (4\mu_4^3 + 27\mu_6^2)f^6 + 18\mu_4\mu_6 f^4 - \mu_4^2 f^2 - 4\mu_6 = 0. \tag{27}$$

2.5. **The elliptic curve in the standard Weierstrass form.**

An elliptic curve in the *standard Weierstrass form* is given by the equation

$$y^2 = 4x^3 - g_2 x - g_3. \tag{28}$$

There is the classical Weierstrass function $\sigma(u) = \sigma(u; g_2, g_3)$ related with this curve (see [35]). It is an entire odd function of $u \in \mathbb{C}$ such that $\sigma(u) = u + (u^5)$. It is a quasiperiodic function with the periods $2\omega_1$, $2\omega_2$:

$$\sigma(u + 2\omega_k) = -\exp\left(2\eta_k(u + \omega_k)\right)\sigma(u), \quad k = 1, 2.$$

The Weierstrass functions $\zeta(u) = \zeta(u; g_2, g_3)$ and $\wp(u) = \wp(u; g_2, g_3)$ are defined by the equations

$$\zeta(u) = (\ln \sigma(u))' \quad \text{and} \quad \wp(u) = -\zeta(u)'. \tag{29}$$

We have $\lim_{u\to 0}(\wp(u) - \frac{1}{u^2}) = 0$. The map $u \mapsto (x, y) = (\wp(u), \wp'(u))$ gives the *Weierstrass uniformization*

$$\wp(u)'^2 = 4\wp(u)^3 - g_2\wp(u) - g_3 \tag{30}$$

of the elliptic curve in the standard Weierstrass form.

In the case $\mu_1 = \mu_2 = \mu_3 = 0$ the elliptic curve in Tate coordinates (8) is given by the equation

$$s = t^3 + \mu_4 t s^2 + \mu_6 s^3. \tag{31}$$

It follows from (21) that the formal group law takes the form

$$F_\mu(t_1,t_2) = t_1 + t_2 + bm\frac{2\mu_4 + 3\mu_6 m}{1 + \mu_4 m^2 + \mu_6 m^3} \tag{32}$$

and therefore

$$\frac{\partial F_\mu(t,t_2)}{\partial t_2}|_{t_2=0} = \rho(t), \quad \text{where} \quad \rho(t) = 1 - 2\mu_4 t s(t) - 3\mu_6 s(t)^2.$$

Using (2), we get

$$f'(u) = \rho(f(u)). \tag{33}$$

In the equation (6) for $x = \widetilde{x}$, $2y = \widetilde{y}$, $(\mu_1,\mu_2,\mu_3) = (0,0,0)$, $-4\mu_4 = g_2$, $-4\mu_6 = g_3$, we obtain the standard Weierstrass equation

$$\widetilde{y}^2 = 4\widetilde{x}^3 - g_2\widetilde{x} - g_3.$$

It is equivalent to the equation (31) with $t = -2\widetilde{x}/\widetilde{y}$ and $s = -2/\widetilde{y}$.

Using the Weierstrass uniformization (30) of (28), we get the uniformization $u \mapsto (t = \frac{-2\wp(u)}{\wp'(u)}, s = \frac{-2}{\wp'(u)})$ of (31).

Lemma 12. *The function*

$$f(u) = \frac{-2\wp(u;g_2,g_3)}{\wp'(u;g_2,g_3)} \tag{34}$$

is the exponential of the formal group (32), where $g_2 = -4\mu_4$ and $g_3 = -4\mu_6$.

Thus the Weierstrass uniformization induces a strong isomorphism of the linear group with the coordinate u and the formal group (32), corresponding to the elliptic curve with Tate coordinates for $\mu_1 = \mu_2 = \mu_3 = 0$.

Proof. We have $t(u) = \frac{-2\wp(u;g_2,g_3)}{\wp'(u;g_2,g_3)}$, $t(0) = 0$, $t'(0) = 1$. Using $s(u) = \frac{-2}{\wp'(u)}$ and (30), we see that $t(u)$ gives a solution of the equation (33), thus $t(u)$ is the exponential of the formal group (32).

Corollary 13. *We have*

$$\frac{-2\wp(u+v;g_2,g_3)}{\wp'(u+v;g_2,g_3)} = F_g(\frac{-2\wp(u;g_2,g_3)}{\wp'(u;g_2,g_3)}, \frac{-2\wp(v;g_2,g_3)}{\wp'(v;g_2,g_3)}),$$

where

$$F_g(t_1,t_2) = t_1 + t_2 - bm\frac{2g_2 + 3g_3 m}{4 - g_2 m^2 - g_3 m^3}. \tag{35}$$

2.6. **The reduction to the standard Weierstrass curve.**

Consider the following linear transformation of $\mathbb{C}P^2$

$$(X:Y:Z) \mapsto (\widetilde{X}:\widetilde{Y}:\widetilde{Z}) = (X + \frac{1}{12}(4\mu_2 + \mu_1^2)Z : 2Y + \mu_1 X + \mu_3 Z : Z).$$

It brings the curve $\mathcal{V}$ to the curve $\widetilde{\mathcal{V}}$ given by the equation

$$\widetilde{Y}^2\widetilde{Z} = 4\widetilde{X}^3 - g_2\widetilde{X}\widetilde{Z}^2 - g_3\widetilde{Z}^3,$$

where

$$g_2 = \frac{1}{12}(4\mu_2 + \mu_1^2)^2 - 2(\mu_1\mu_3 + 2\mu_4), \tag{36}$$

$$g_3 = \frac{1}{6}(\mu_1\mu_3 + 2\mu_4)(4\mu_2 + \mu_1^2) - \frac{1}{6^3}(4\mu_2 + \mu_1^2)^3 - 4\mu_6 - \mu_3^2. \tag{37}$$

Notice that this linear transformation brings O to O, so it gives a homomorphism of formal groups $F_\mu \mapsto F_g$, where F_g is the formal group (35) given by the geometric structure of the group on the elliptic curve $\widetilde{\mathcal{V}}$.

In the coordinate map $Y \neq 0$ we have the Tate coordinates $\widetilde{t} = -2\widetilde{X}/\widetilde{Y}$, $\widetilde{s} = -2\widetilde{Z}/\widetilde{Y}$, and the curve $\widetilde{\mathcal{V}}$ is given by the equation

$$\widetilde{s} = \widetilde{t}^3 - \frac{1}{4}g_2\widetilde{t}\widetilde{s} - \frac{1}{4}g_3\widetilde{s}^3.$$

We come to equation (31) for $g_2 = -4\mu_4$ and $g_3 = -4\mu_6$.

Lemma 14. *The Tate coordinates t and s of the curve $\mathcal{V}$ are connected to the Tate coordinates $\widetilde{t}$ and $\widetilde{s}$ of the curve $\widetilde{\mathcal{V}}$ by the following formulas:*

$$t = \frac{\widetilde{t} - \frac{1}{12}(4\mu_2 + \mu_1^2)\widetilde{s}}{1 + \frac{1}{2}\mu_1\widetilde{t} - \frac{1}{2}(\frac{1}{12}\mu_1(4\mu_2 + \mu_1^2) - \mu_3)\widetilde{s}}, \quad s = \frac{\widetilde{s}}{1 + \frac{1}{2}\mu_1\widetilde{t} - \frac{1}{2}(\frac{1}{12}\mu_1(4\mu_2 + \mu_1^2) - \mu_3)\widetilde{s}}.$$

Let

$$\psi(t) = \widetilde{t}(t) = \frac{t + \frac{1}{12}(4\mu_2 + \mu_1^2)s(t)}{1 - \frac{1}{2}\mu_1 t - \frac{1}{2}\mu_3 s(t)}.$$

Thus

$$\psi(F_\mu(t_1, t_2)) = F_g(\psi(t_1), \psi(t_2)). \tag{38}$$

In the coordinate map $Z \neq 0$ we have

$$\widetilde{x} = x + \frac{1}{12}(4\mu_2 + \mu_1^2), \qquad \widetilde{y} = 2y + \mu_1 x + \mu_3. \tag{39}$$

The curve $\widetilde{\mathcal{V}}$ is given by the equation

$$\widetilde{y}^2 = 4\widetilde{x}^3 - g_2\widetilde{x} - g_3.$$

Using the Weierstrass uniformization of the curve $\widetilde{\mathcal{V}}$: $(\widetilde{x}, \widetilde{y}) = (\wp(u; g_2, g_3), \wp'(u, g_2, g_3))$, we obtain the uniformization of the curve $\mathcal{V}$:

$$x = \wp(u; g_2, g_3) - \frac{1}{12}(4\mu_2 + \mu_1^2), \quad y = \frac{1}{2}(\wp'(u, g_2, g_3) - \mu_1\wp(u; g_2, g_3) + \frac{1}{12}\mu_1(4\mu_2 + \mu_1^2) - \mu_3).$$

Using (34) and (38), we get:

Corollary 15. *The exponential of the general elliptic formal group (17) is*

$$f(u) = -2\frac{\wp(u; g_2, g_3) - \frac{1}{12}(4\mu_2 + \mu_1^2)}{\wp'(u; g_2, g_3) - \mu_1\wp(u; g_2, g_3) + \frac{1}{12}\mu_1(4\mu_2 + \mu_1^2) - \mu_3}, \tag{40}$$

where g_2 and g_3 are given by (36) and (37).

Thus the Weierstrass uniformization of the curve (28) induces a strong isomorphism of the linear group with the coordinate u and the formal group (17), corresponding to the elliptic curve with Tate coordinates (8).

Corollary 16. *The solution $f(u)$ of the equation (24) with conditions $f(0) = 0$, $f'(0) = 1$ is given by formula (40).*

Corollary 17. $f(u) \in HE[[u]]$.

Corollary 18. *We have*

$$\frac{1}{f(u)} = -\frac{1}{2}\frac{\wp'(u) + \wp'(w)}{\wp(u) - \wp(v)} + \frac{\mu_1}{2}, \tag{41}$$

where $\wp(u) = \wp(u; g_2, g_3)$, $\wp'(w) = -\mu_3$ and $\wp(v) = \frac{1}{12}(4\mu_2 + \mu_1^2)$.

Corollary 19. *The exponential of the elliptic formal group law in the non-degenerate case is the ellipitc function of order 2 iff $\mu_6 = 0$. It is the ellipitc function of order 3 in the general non-degenerate case.*

We will consider the elliptic sine $f(u) = sn(u; \delta, \varepsilon)$ as the solution of the equation

$$f(u)'^2 = R(f(u)), \quad R(t) = 1 - 2\delta t^2 + \varepsilon t^4$$

with conditions $f(0) = 0$, $f'(0) = 1$. It has the classical addition law

$$F(t_1, t_2) = \frac{t_1\sqrt{R(t_2)} + t_2\sqrt{R(t_1)}}{1 - \varepsilon t_1^2 t_2^2} \tag{42}$$

Example 20. The exponential of the elliptic formal group law is the elliptic sine with parametra (δ, ε) if and only if $(\mu_1, \mu_3, \mu_6) = (0, 0, 0)$, $\mu_2 = \delta$, $\mu_4 = \frac{1}{4}(\delta^2 - \varepsilon)$. Thus the classical addition law (42) gives a formal group over $\mathbb{Z}[\mu_2, \mu_4]$.

2.7. **The 2-height of the elliptic formal group laws.**

For any formal group $F(t_1, t_2)$ over A the formula

$$F(t, t) = t^{2^h} + ... (\mathrm{mod}\, 2)$$

holds for some $h \geq 1$. Such number h is called the *2-height* of the formal group F.

Let us find h for the elliptic formal group laws.

Over the ring $\mathbb{Z}_2[\mu_i][[t]]$ for $t_1 = t_2 = t$ we have

$$m = s'(t) = \frac{t^2 + \mu_1 s(t) + \mu_4 s(t^2)}{1 + \mu_1 t + \mu_2 t^2 + \mu_6 s(t^2)}, \quad b = ts'(t) + s(t) = \frac{\mu_1 t s(t) + \mu_3 s(t^2)}{1 + \mu_1 t + \mu_2 t^2 + \mu_6 s(t^2)},$$

$$n = s'(t) + t^2 \frac{(1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3)}{(1 + \mu_3 b + \mu_6 b^2)}.$$

Lemma 21. *Over the ring $\mathbb{Z}_2[\mu_i]$ we have*

$$F_\mu(t, t) = (\mu_1 t^2 + \mu_3 t^2 m + \mu_4 t^2 b)\frac{(1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3)}{(1 + \mu_2 n + \mu_4 n^2 + \mu_6 n^3)(1 - \mu_3 b - \mu_6 b^2)^2}. \tag{43}$$

Thus, we have:

Corollary 22. *For $\mu_1 \neq 0$ the height is* 1.

For $\mu_1 = 0$, the formula (43) takes the form

$$F_\mu(t,t) = \frac{\mu_3 t^4}{1 - \mu_2 t^2 - \mu_6 s(t^2)} \frac{(1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3)}{(1 + \mu_2 n + \mu_4 n^2 + \mu_6 n^3)(1 - \mu_3 b - \mu_6 b^2)^2}.$$

Thus in the case $\mu_1 = 0$, $\mu_3 \neq 0$ the height is 2 and the elliptic curve is supersingular.

For $\mu_1 = 0$, $\mu_3 = 0$ we see the height is ∞.

2.8. **The formal group law over the ring with trivial multiplication.**

We will describe the formal group law modulo the ideal of decomposable element in the ring E in order to get important information on the homomorphism $\phi : \mathcal{A} \to E$.

Let $E^{(1)} = E/(\widetilde{E})^2$, where $\widetilde{E} = Ker(E \mapsto \mathbb{Z} : \mu_i \mapsto 0)$. Over the ring $E^{(1)}$ from (8) we have

$$s = t^3 + \mu_1 t^4 + \mu_2 t^5 + \mu_3 t^6 + \mu_4 t^7 + \mu_6 t^9.$$

From (17) we get

$$F_\mu(t_1,t_2) = t_1 + t_2 - t_1 t_2 \left[\mu_1 + \mu_2(t_1+t_2) + \mu_3(2t_1^2 + 3t_1t_2 + 2t_2^2) + \right.$$
$$\left. + 2\mu_4(t_1+t_2)(t_1^2 + t_1t_2 + t_2^2) + 3\mu_6(t_1+t_2)(t_1^2 + t_1t_2 + t_2^2)^2\right].$$

It follows from (2) that

$$f'(t) = \frac{\partial F_\mu(t_1,t_2)}{\partial t_2}|_{t_2=0} = 1 - (\mu_1 t_1 + \mu_2 t_1^2 + 2\mu_3 t_1^3 + 2\mu_4 t_1^4 + 3\mu_6 t_1^6).$$

Thus, over $E^{(1)}$ the formula holds:

$$f(t) = t - \frac{1}{2}\mu_1 t^2 - \frac{1}{3}\mu_2 t^3 - \frac{1}{2}\mu_3 t^4 - \frac{2}{5}\mu_4 t^5 - \frac{3}{7}\mu_6 t^7.$$

We get $f_1 = -\frac{1}{2}\mu_1$, $f_2 = -\frac{1}{3}\mu_2$, $f_3 = -\frac{1}{2}\mu_3$, $f_4 = -\frac{2}{5}\mu_4$, $f_6 = -\frac{3}{7}\mu_6$. Notice that $\nu(2) = 2$, $\nu(3) = 3$, $\nu(4) = 2$, $\nu(5) = 5$, $\nu(7) = 7$.

Using the description of the multiplicative generators of the ring $\mathcal{A}$ (see page 4), we obtain

$$a_n^* = \nu(n+1) b_n^* \quad \text{and} \quad \phi(b_n) = f_n.$$

Let E_F be the subring of E, generated by the coefficients of F. Therefore

1. The composition of the maps $E_F \hookrightarrow E$ and $E \to \mathbb{Z}[\mu_1,\mu_2,\mu_3]$: $\mu_i \mapsto \mu_i$, $i = 1,2,3$, $\mu_i \mapsto 0$, $i = 4,6$ is an epimorphism.
2. The composition of the maps $(E_F)_{(p)} \hookrightarrow E_{(p)}$ and $E_{(p)} \to \mathbb{Z}_{(p)}[\mu_1,\mu_2,\mu_3,\mu_6]$: $\mu_i \mapsto \mu_i$, $i = 1,2,3,6$, $\mu_4 \mapsto 0$, is an epimorphism if $p \neq 3$.
3. The composition of the maps $(E_F)_{(p)} \hookrightarrow E_{(p)}$ and $E_{(p)} \to \mathbb{Z}_{(p)}[\mu_1,\mu_2,\mu_3,\mu_4]$: $\mu_i \mapsto \mu_i$, $i = 1,2,3,4$, $\mu_6 \mapsto 0$, is an epimorphism if $p \neq 2$.

In particular, the coefficients of the formal group law $F(t_1,t_2)$ generate multiplicatively the ring $E_{(p)}$ for $p \neq 2,3$.

2.9. **Authomorphisms of elliptic formal group laws.**

It is wellknown that the group structure on the elliptic curve can have non-trivial automorphisms of order only $2, 3, 4$, and 6. We will derive this result using the formal group law in Tate coordinates. We included the exposition of this result because it will be used below.

Let us consider the formal group law F_μ over $E \otimes (\mathbb{Z}[\alpha]/J)$, where J is some ideal.

Any linear authomorphism of a formal group law is given by the identity

$$F_\mu(\alpha t_1, \alpha t_2) = \alpha F_\mu(t_1, t_2). \tag{44}$$

Using (2) we obtain

$$g'(\alpha t) = g'(t).$$

Thus $g(\alpha t) = \alpha g(t)$ and $f(\alpha t) = \alpha f(t)$.

Because $g(t) = t + ...$ is a series of t, it can be presented in the form $g(t) = t\psi(t^n)$ for some n and some series $\psi(t)$.

In the case $\psi(t^n) = 1$ we get $g(t) = t$, so $F_\mu(t_1, t_2) = t_1 + t_2$ is a linear group and the identity (44) is valid for any ideal J.

For $\psi \neq 1$ fix the maximal n in the form $g(t) = t\psi(t^n)$. Then it follows from the identity $g(\alpha t) = \alpha g(t)$ that $(\alpha^n - 1) \in J$. Thus $\frac{1}{g'(t)}$ can be presented in the form $\phi(t^n)$ for some series $\phi(t)$.

Let us consider the case of elliptic formal group laws.

Using (22) and (2), we get the condition $\rho(\alpha t) = \rho(t)$ for

$$\rho(t) = 1 - \mu_1 t - \mu_2 t^2 - 2\mu_3 s - 2\mu_4 ts - 3\mu_6 s^2. \tag{45}$$

Let $n = 2$. The function $\rho(t) = \phi(t^2)$ should be an even function of t, thus $\mu_1 = \mu_3 = 0$. In this case

$$\rho(t) = 1 - \mu_2 t^2 - 2\mu_4 ts - 3\mu_6 s^2,$$

where $s(t)$ determined by the relation $s = t^3 + \mu_2 t^2 s + \mu_4 ts^2 + \mu_6 s^3$ is an odd function. See Example 57.

Let $n = 3$. Then $\rho(t) = \phi(t^3)$, thus $\mu_1 = \mu_2 = \mu_4 = 0$. In this case

$$\rho(t) = 1 - 2\mu_3 s - 3\mu_6 s^2,$$

where $s(t)$ determined by the relation $s = t^3 + \mu_3 s^2 + \mu_6 s^3$, thus s is a function of t^3. See Example 60.

Let $n = 4$. Then $\rho(t) = \phi(t^4)$, thus $\mu_1 = \mu_2 = \mu_3 = \mu_6 = 0$. In this case

$$\rho(t) = 1 - 2\mu_4 ts,$$

where $ts(t)$ determined by the relation $ts = t^4 + \mu_4 t^2 s^2$, thus ts is a function of t^4. See Example 61.

Let $n = 6$. Then $\frac{1}{g'(t)} = \phi(t^6)$, thus $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$. In this case

$$\rho(t) = 1 - 3\mu_6 s^2,$$

where $s(t)$ determined by the relation $s = t^3 + \mu_6 s^3$, thus $s(t)$ is an odd function of t^3 and $s^2(t)$ is a function of t^6. See Example 62.

Let $n = 5$ or $n \geq 7$. Then we should have $\rho(t) = \phi(t^n)$, but it follows from (45) that $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_6 = 0$, thus $g'(t) = 1$ and $\psi(t^n) = 1$.

2.10. **Differential equations, connecting the Tate coordinates of the elliptic curve with its parameters.**

Using the relation (8)

$$s = t^3 + \mu_1 ts + \mu_2 t^2 s + \mu_3 s^2 + \mu_4 ts^2 + \mu_6 s^3,$$

one can consider s as a function $s(t;\mu)$ of t and $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_6)$. Then

$$(1 - \mu_1 t - \mu_2 t^2 - 2\mu_3 s - 2\mu_4 ts - 3\mu_6 s^2)\frac{\partial s}{\partial t} = 3t^2 + \mu_1 s + 2\mu_2 ts + \mu_4 s^2. \tag{46}$$

Lemma 23. *Let $\mu(v) = (0, 3v + c_2, c_3, 3v^2 + 2c_2 v + c_4, v^3 + c_2 v^2 + c_4 v + c_6)$, where c_k do not depend on v, $k = 2, 3, 4, 6$. The function $S(t, v) = s(t, \mu(v))$ satisfies the Hopf equation*

$$\frac{\partial S}{\partial v} = S\frac{\partial S}{\partial t} \tag{47}$$

with the initial conditions $S(t, 0) = s_0(t)$, where $s_0(t)$ is defined by the equation

$$s_0 = t^3 + c_2 t^2 s_0 + c_3 s_0^2 + c_4 t s_0^2 + c_6 s_0^3.$$

Proof. Consider the path $\mu : \mathbb{C} \to \mathbb{C}^5$: $v \mapsto \mu(v)$, where $\mu_i(v)$ are given in the lemma. Thus

$$\frac{\partial s}{\partial v}(1 - \mu_1 t - \mu_2 t^2 - 2\mu_3 s - 2\mu_4 ts - 3\mu_6 s^2) = 3t^2 s + 2(3v + c_2)ts^2 + (3v^2 + 2c_2 v + c_4)s^3.$$

Comparing with (46), we come to the Hopf equation (47).

Note: Let $U = U(\tau, v; \alpha)$ be the solution of the Hopf equation

$$\frac{\partial U}{\partial v} = U\frac{\partial}{\partial \tau}U$$

with $U(\tau, 0; \alpha) = \frac{\tau^2}{1-\alpha\tau}$.

Then $U(\tau, v; \alpha) = \sum\limits_{n\geqslant 0} f(As^n)\tau^{n+2}$ where $f(As^n) = \alpha^n + f_{n-1}\alpha^{n-1}v + \ldots + f_0 v^n$. Here As^n is n-dimensional associahedron, Stasheff polytope K_{n+2}, and $f_k = f_k(As^n)$ is the number of k-dimensional faces. Then the function U satisfies the equation (see [4])

$$v(\alpha + v)U^2 - \big(1 - (\alpha + 2v)\tau\big)U + \tau^2 = 0.$$

Lemma 24. *The path $\mu(v)$, where $\mu_1(v) = 0$, $\mu_2(v) = 3v + c_2$, $\mu_3(v) = c_3$, $\mu_4(v) = 3v^2 + 2c_2 v + c_4$, $\mu_6(v) = v^3 + c_2 v^2 + c_4 v + c_6$, defines a family of elliptic curves with the same standard Weirstrass form for any v.*

Proof. The reduction to the standard Weierstrass form gives a mapping $\mathbb{C}^5 \to \mathbb{C}^2 : \mu \mapsto (g_2, g_3)$, defined by (36), (37).
Direct calculations give $g_2(\mu(v)) = g_2(\mu(0))$ and $g_3(\mu(v)) = g_3(\mu(0))$.

Lemma 25. *The function* $S(\tau, v) = s(t, \mu(v))$, *where* $\tau = t^3$, $\mu_1(v) = \mu_2(v) = \mu_4(v) = 0$, $\mu_3(v) = \alpha v + c_3$, $\mu_6(v) = \beta v + c_6$, *satisfies the equation*

$$\frac{\partial S}{\partial v} = (\alpha S^2 + \beta S^3)\frac{\partial S}{\partial \tau} \tag{48}$$

with the initial conditions $S(\tau, 0) = s_0(\tau)$, *where* $s_0(\tau)$ *is defined by the equation*

$$s_0 = \tau + c_3 s_0^2 + c_6 s_0^3.$$

Proof. Consider the path $\mu : \mathbb{C} \to \mathbb{C}^5$: $v \mapsto \mu(v)$, where $\mu_i(v)$ are given in the lemma. We have

$$s = t^3 + \mu_3 s^2 + \mu_6 s^3,$$

so $s(t, \mu(v))$ depends on $\tau = t^3$. Thus for $S(\tau, v)$ we have

$$\frac{\partial S}{\partial \tau}(1 - 2\mu_3 S - 3\mu_6 S^2) = 1,$$

$$\frac{\partial S}{\partial v}(1 - 2\mu_3 S - 3\mu_6 S^2) = \alpha S^2 + \beta S^3,$$

and we come to the equation (48).

3. **Hurwitz series defined by elliptic curves.**

3.1. **The sigma-function of the elliptic curve.** The sigma function $\sigma(u)$ has a series expansion in powers of u over the polynomial ring $\mathbb{Q}[g_2, g_3]$ in the vicinity of $u = 0$. An initial segment of the series has the form

$$\sigma(u) = u - \frac{g_2\, u^5}{2 \cdot 5!} - \frac{6\, g_3\, u^7}{7!} - \frac{g_2^2\, u^9}{4 \cdot 8!} - \frac{18\, g_2\, g_3\, u^{11}}{11!} + (u^{13}). \tag{49}$$

Theorem 26. *The sigma function* $\sigma(u)$ *is a Hurwitz series over* $\mathbb{Z}[\frac{1}{2}][g_2, g_3]$.

For the proof see [27].

The following operators annihilate the sigma function

$$Q_0 = 4g_2\frac{\partial}{\partial g_2} + 6g_3\frac{\partial}{\partial g_3} - u\frac{\partial}{\partial u} + 1, \quad Q_2 = 6g_3\frac{\partial}{\partial g_2} + \frac{1}{3}g_2^2\frac{\partial}{\partial g_3} - \frac{1}{2}\frac{\partial^2}{\partial u^2} - \frac{1}{24}g_2 u^2.$$

Theorem 27. *Set*

$$\sigma(u) = u \sum_{i,j \geq 0} \frac{a_{i,j}}{(4i + 6j + 1)!}(\frac{g_2 u^4}{2})^i (2g_3 u^6)^j.$$

The Weierstrass recursion for the sigma function is given by the formulas

$$a_{i,j} = 3(i+1)a_{i+1,j-1} + \frac{16}{3}(j+1)a_{i-2,j+1} - \frac{1}{3}(4i+6j-1)(2i+3j-1)a_{i-1,j} \text{ for } i \geq 0,\ j \geq 0,\ (i, j) \neq (0, 0),$$

$$a_{0,0} = 1, \quad a_{i,j} = 0 \text{ for } i < 0 \text{ or } j < 0,$$

which define $a_{i,j}$ *for* $2i + 3j \leq 0$, *and if* $a_{i,j}$ *is defined for* $2i + 3j < m$ *where* $m > 0$, *then* $a_{i,j}$ *is defined recursively for* $2i + 3j = m$.

Proof. Consider the equation $Q_2\sigma(u) = 0$. It follows that

$$18(i+1)\sum_{i\geq 0, j\geq 1}\frac{a_{i+1,j-1}}{(4i+6j-1)!}(\frac{g_2u^4}{2})^i(2g_3u^6)^j + 32(j+1)\sum_{i\geq 2, j\geq 0}\frac{a_{i-2,j+1}}{(4i+6j-1)!}(\frac{g_2u^4}{2})^i(2g_3u^6)^j -$$

$$-6(4i+6j+1)(4i+6j)\sum_{i\geq 0, j\geq 0}\frac{a_{i,j}}{(4i+6j+1)!}(\frac{g_2u^4}{2})^i(2g_3u^6)^j - \sum_{i\geq 1, j\geq 0}\frac{a_{i-1,j}}{(4i+6j-3)!}(\frac{g_2u^4}{2})^i(2g_3u^6)^j = 0.$$

For $i = j = 0$ we get $0 = 0$,

for $j = 0, i = 1$: $\quad a_{1,0} = -a_{0,0}$,

for $i = 0, j \geq 1$: $\quad a_{0,j} = 3a_{1,j-1}$,

for $j = 0, i \geq 2$: $\quad 3a_{i,0} = 16a_{i-2,1} - (2i-1)(4i-1)a_{i-1,0}$,

for $i = 1, j \geq 1$: $\quad a_{1,j} = 6a_{2,j-1} - (2j+1)(3j+1)a_{0,j}$,

for $i \geq 2, j \geq 1$: $\quad 3a_{i,j} = 9(i+1)a_{i+1,j-1} + 16(j+1)a_{i-2,j+1} - (4i+6j-1)(2i+3j-1)a_{i-1,j}$.

Put $a_{i,j} = 0$ for $i < 0$ or $j < 0$. For all $(i,j) \neq (0,0)$ the formula holds:

$$a_{i,j} = 3(i+1)a_{i+1,j-1} + \frac{16}{3}(j+1)a_{i-2,j+1} - \frac{1}{3}(4i+6j-1)(2i+3j-1)a_{i-1,j}.$$

The definition of the sigma function gives the initial condition for the recursion $a_{0,0} = 1$.

Thus $a_{i,j} \in \mathbb{Z}[\frac{1}{3}]$ and we obtain

Corollary 28. *The sigma function is a Hurwitz series over* $\mathbb{Z}[\frac{1}{3}, \frac{g_2}{2}, 2g_3]$*:*

$$\sigma(u) \in H\mathbb{Z}[\frac{1}{3}, \frac{g_2}{2}, 2g_3][[u]].$$

From $\sigma(u) \in H\mathbb{Z}[\frac{1}{3}, \frac{g_2}{2}, 2g_3][[u]]$ and $\sigma(u) \in H\mathbb{Z}[\frac{1}{2}, g_2, g_3][[u]]$ we obtain:

Theorem 29. *The sigma function is a Hurwitz series over* $\mathbb{Z}[\frac{g_2}{2}, 2g_3]$*:*

$$\sigma(u) \in H\mathbb{Z}[\frac{g_2}{2}, 2g_3][[u]],$$

that is $a_{i,j} \in \mathbb{Z}$.

Explicitly, we have

$$a_{0,0} = 1,\ a_{1,0} = -1,\ a_{2,0} = -3^2,\ a_{3,0} = 3\cdot 23,\ a_{4,0} = 3\cdot 107,$$

$$a_{0,1} = -3,\ a_{1,1} = -2\cdot 3^2,\ a_{2,1} = 3^3\cdot 19,\ a_{3,1} = 2^2\cdot 3^3\cdot 311,\ a_{4,1} = 3^3\cdot 5\cdot 20807,$$

$$a_{0,2} = -2\cdot 3^3,\ a_{1,2} = 2^3\cdot 3^3\cdot 23,\ a_{2,2} = 2^2\cdot 3^5\cdot 5\cdot 53,\ a_{3,2} = 2^3\cdot 3^4\cdot 5\cdot 37\cdot 167,\ a_{4,2} = -2\cdot 3^6\cdot 5\cdot 17\cdot 3037.$$

Let $b_{i,j} = 2^{3i+4j}\, 3^{i+j}\, \frac{i!\, j!}{(4i+6j+1)!}\, a_{i,j}$; $b_{i,j} \in \mathbb{Q}$. We get

$$(4i+6j+1)(2i+3j)b_{i,j} = 3jb_{i+1,j-1} - 2ib_{i-1,j} + 32i(i-1)b_{i-2,j+1}.$$

Computer calculations show that for $i + j \leq 100$ such $p_{i,j} \in \mathbb{Z}$ and $q_{i,j} \in \mathbb{Z}$ exist that $b_{i,j} = \frac{p_{i,j}}{q_{i,j}}$, $p_{i,j}$ is coprime with 6 and $q_{i,j}$ is coprime with 6.

This leads us to the following conjecture:

Conjecture. Let $a(i,j) = 2^k 3^l s(i,j)$, where $s(i,j) \in \mathbb{Z}$ is coprime with 2 and 3. Let
$$\frac{(4i+6j+1)!}{2^{3i+4j}\,3^{i+j}\,i!\,j!} = 2^{k_1}3^{l_1}s_1(i,j), \text{ where } s_1(i,j) \in \mathbb{Z} \text{ is coprime with 2 and 3.}$$
Then $k = k_1$, $l = l_1$.

By the definition $\zeta(u) = (\ln\sigma(u))'$ and $\wp(u) = -\zeta(u)'$, therefore
$$\zeta(u) = \frac{\sigma(u)'}{\sigma(u)}, \quad \wp(u) = \frac{\sigma(u)'^2 - \sigma(u)''\sigma(u)}{\sigma(u)^2}.$$
Using that $\sigma'(0) = 1$, we see that $\frac{1}{\zeta(u)}$ and $\frac{1}{\wp(u)}$ are Hurwitz series over $\mathbb{Z}[\frac{g_2}{2}, 2g_3]$.

Corollary 30. *For any v let $a_2 = \wp(v)$, $a_3 = \wp'(v)$, $a_4 = \frac{g_2}{2}$. Then $g_3 = -a_3^2+4a_2^3-2a_2a_4$. Thus $\sigma(u) \in H\mathbb{Z}[a_2,a_3,a_4][[u]]$.*

3.2. **The Baker-Akhiezer function of the elliptic curve.** The Baker-Akhiezer function plays an important role in the modern theory of integrable systems, see [17]. In this work I. M. Krichever introduced the addition theorem for this function and demonstrated its important applications.

Consider *the Lame equation*
$$\Phi''(u) - 2\wp(u)\Phi(u) = \wp(v)\Phi(u). \tag{50}$$
The quasiperiodic solutions of (50) such that $\lim\limits_{u\to 0}\left(\Phi(u) - \frac{1}{u}\right) = 0$ are $\Phi(u) = \Phi(u;v)$ and $\Phi_1(u) = \Phi(u;-v)$, where
$$\Phi(u;v) = \frac{\sigma(v-u)}{\sigma(u)\sigma(v)}\exp(\zeta(v)u) \tag{51}$$
is *the Baker-Akhiezer function.* The periodic properties are
$$\Phi(u+2\omega_k;v) = \Phi(u;v)\exp(2\zeta(v)\omega_k - 2\eta_k v), \tag{52}$$
$$\Phi(u;v+2\omega_k) = \Phi(u;v). \tag{53}$$
Thus $(\ln\Phi(u))'$ is a doubly periodic meromorphic function of u (see [35]):
$$(\ln\Phi(u))' = \zeta(u-v) + \zeta(v) - \zeta(u) = \frac{1}{2}\frac{\wp'(u)+\wp'(v)}{\wp(u)-\wp(v)}. \tag{54}$$
The function $\Phi(u;v)$ considered as a function of v has an exponential singularity in $v = 0$.

Remark 31. The theory of algebrogeometric solutions of integrable equations like KdV
$$\frac{\partial V}{\partial t} = V''' - 6VV' \tag{55}$$
started from the work of S. P. Novikov [23]. Consider the operators
$$L_1 = \frac{d^2}{du^2} - 2\wp(u), \tag{56}$$
$$L_2 = -2\frac{d^3}{du^3} + 6\wp(u)\frac{d}{du} + 3\wp'(u). \tag{57}$$

The operators L_1 and L_2 are the Lax pair of the Schrodinger operator L_1 with the potential $V(u) = 2\wp(u)$. The function $2\wp(u)$ is the solution of the stationary KdV equation. Thus the operators L_1 and L_2 commute. We have

$$L_1\Phi(u) = \wp(v)\Phi(u),$$

$$L_2\Phi(u) = \wp'(v)\Phi(u).$$

Thus the fucnction $\Phi(u)$ is the common eigenfunction of the operators L_1 and L_2 and the pair of eigenvalues $\big(\wp(v), \wp'(v)\big)$ defines a point on the Weierstrass curve.

Set $a_2 = \wp(v),\ a_3 = \wp'(v),\ a_4 = \frac{1}{2}g_2$ for the given v.

Theorem 32. *In the vicinity of $u = 0$ the function $f_0(u) = 1/\Phi(u)$ is a Hurwitz series over $\mathbb{Z}[a_2, a_3, a_4]$.*

The function $f_0(u)$ is regular in the vicinity of $u = 0$. We have

$$f_0(u) = f_0(u, v) = \sigma(u) \exp \psi(u, v), \tag{58}$$

where

$$\psi(u, v) = \ln \sigma(v) - \ln \sigma(v - u) - \zeta(v)u.$$

The proof of Theorem 32 is based on the following lemma:

Lemma 33. *In the vicinity of $u = 0$ the function $\psi(u, v)$ is a Hurwitz series over $\mathbb{Z}[a_2, a_3, a_4]$.*

Notice that the functions $\wp(v),\ \wp'(v),\ \frac{1}{2}g_2$ are algebraically independent in the general case, so we can consider $\psi(u, v)$ over a ring of algebraically independent variables.

Proof of Lemma 33. Teylor decomposition of the function $\psi(u, v)$ at $u = 0$ is given by

$$\psi(u, v) = \sum_{k=2}^{\infty} (-1)^k \Big(- \frac{d^k \ln \sigma(v)}{dv^k} \Big) \frac{u^k}{k!} = \sum_{k=1}^{\infty} (-1)^{k-1} \wp^{(k-1)}(v) \frac{u^{k+1}}{(k+1)!}. \tag{59}$$

Using Weierstrass's uniformization of the elliptic curve (30), we get

$$\wp''(v) = 6\wp(v)^2 - g_2/2.$$

Thus

$$\wp^{(k)}(v) = p_{k+2}(a_2, a_3, a_4),\ k \geqslant 0,$$

where

$$\wp^{(0)}(v) = \wp(v) = a_2, \quad \wp^{(1)}(v) = \frac{\partial}{\partial v}\wp(v) = a_3.$$

It follows that $p_2(a_2, a_3, a_4) = a_2$ and

$$p_{k+1}(a_2, a_3, a_4) = \left(a_3 \frac{\partial}{\partial a_2} + (6a_2^2 - a_4) \frac{\partial}{\partial a_3} \right) p_k(a_2, a_3, a_4),\ k \geqslant 2.$$

Thus

$$\psi(u,v)=\sum_{k=2}^{\infty}(-1)^k p_k(a_2,a_3,a_4)\frac{u^k}{k!}, \tag{60}$$

where $p_k(a_2,a_3,a_4)$ is a homogeneous polynomial with integer coefficients of a_2, a_3, a_4.

Remark 34. We have $\deg a_2=-4$, $\deg a_3=-6$, $\deg a_4=-8$. Thus
for even $k\geq 0$ we have $p_k(a_2,a_3,a_4)=r_k(a_2,a_3^2,a_4)$ for some $r_k(a_2,a_3^2,a_4)\in\mathbb{Z}[a_2,a_3^2,a_4]$.
For even $k\geq 0$ we have $p_{k+3}(a_2,a_3,a_4)=a_3\,q_k(a_2,a_3^2,a_4)$ for some
$q_k(a_2,a_3^2,a_4)\in\mathbb{Z}[a_2,a_3^2,a_4]$. We have $q_0(a_2,a_3^2,a_4)=1$.

Corollary 35. *In the vicinity of* $u=0$ *the function* $\exp\psi(u,v)$ *is a Hurwitz series over* $\mathbb{Z}[a_2,a_3,a_4]$.

By Corollary 30 $\sigma(u)\in H\mathbb{Z}[a_2,a_3,a_4][[u]]$. Summarizing this facts, we get the proof of Theorem 32.

3.3. **The generalized Baker-Akhiezer function.**

We will need the following functions for the description of the Krichever genus.

Consider the function

$$\hat{\Phi}(u)=\Phi(u)\exp(-\frac{\mu_1}{2}u), \tag{61}$$

where $\Phi(u)=\Phi(u;v)$ is the Baker-Akhiezer function (51), and the function
$U_1(u)=U_1(u;v)=-\frac{1}{2}\frac{\wp'(u)+\wp'(v)}{\wp(u)-\wp(v)}$.

The function $\hat{\Phi}(u)$ is a solution of the equation

$$\hat{\Phi}''(u)-(2\wp(u)+\mu_1U_1(u))\hat{\Phi}(u)=(\wp(v)+\frac{\mu_1^2}{4})\hat{\Phi}(u). \tag{62}$$

It follows from Theorem 32 that in the vicinity of $u=0$ the function $1/\hat{\Phi}(u)$ is a Hurwitz series over $\mathbb{Z}[a_1,a_2,a_3,a_4]$, where $a_1=\frac{\mu_1}{2},\ a_2=\wp(v),\ a_3=\wp'(v),\ a_4=\frac{1}{2}g_2$.

Definition 36. The generalized Baker-Akhiezer function is $\Psi(u)=\Psi(u;v;\alpha;\mu)$ defined by the formula

$$\Psi(u)=\frac{\sigma(u+v)^{\frac{1}{2}(1-\alpha)}\sigma(v-u)^{\frac{1}{2}(1+\alpha)}}{\sigma(u)\sigma(v)}\exp\left((-\frac{\mu_1}{2}+\alpha\zeta(v))u\right), \tag{63}$$

where $\sigma(u)=\sigma(u;g_2(\mu),g_3(\mu))$, $\zeta(v)=\zeta(v;g_2(\mu),g_3(\mu))$.

We have

$$(\ln\Psi(u))'=\frac{1}{2}\frac{\wp'(u)+\alpha\wp'(v)}{\wp(u)-\wp(v)}-\frac{\mu_1}{2}. \tag{64}$$

Theorem 37. *The function (63) with* $\alpha=\frac{\wp'(w)}{\wp'(v)}$ *satisfies the equation*

$$\Psi''(u)-(2\wp(u)+\mu_1U_1(u)-U_2(u))\Psi(u)=(\wp(v)+\frac{\mu_1^2}{4})\Psi(u) \tag{65}$$

where $U_2(u) = U_2(u; v, w) = U_1(v; u, w)U_1(v; u, -w)$ *and*

$$U_1(u; v, w) = -\frac{1}{2}\frac{\wp'(u) + \wp'(w)}{\wp(u) - \wp(v)}. \tag{66}$$

Note that for $w \to v$ or $w \to -v$ (that is $\mu_6 \to 0$) the equation (65) comes to the equation (62), and for $(\mu_1, \mu_6) \to (0, 0)$ to the Lame equation.

The formula holds

$$\Psi(u; v; \alpha) = \Phi(u; v) \exp(-\frac{\mu_1}{2}u) \left(\frac{\Phi(u; -v)}{\Phi(u; v)}\right)^{\frac{1}{2}(1-\alpha)}.$$

Corollary 38. *The periodic properties are*

$$\Psi(u + 2\omega_k; v; \alpha) = \Psi(u; v; \alpha) \exp(\alpha(2\zeta(v)\omega_k - 2\eta_k v) - \mu_1\omega_k). \tag{67}$$

Let $\alpha = \frac{\wp'(w)}{\wp'(v)}$. Then

$$\left(\frac{\Phi(u; -v)}{\Phi(u; v)}\right)^{\frac{1}{2}\frac{\wp'(v)-\wp'(w)}{\wp'(v)}} = \exp\left(\frac{1}{2}(\wp'(v) - \wp'(w))\frac{\ln(\sigma(u + v)) - \ln(\sigma(v - u)) - 2\zeta(v)u}{\wp'(v)}\right).$$

Let ω_k, $k = 1, 2, 3$ be the half-periods of the elliptic curve in the standard Weierstrass form (28), where $\omega_1 + \omega_2 + \omega_3 = 0$.

We have

$$\Phi(u; \omega_k) = \Phi(u; -\omega_k) = \frac{\sigma(\omega_k - u)}{\sigma(u)\sigma(\omega_k)} \exp(\eta_k u).$$

Thus $\Phi(u; \omega_k)^2 = \wp(u) - e_k$, where $e_k = \wp(\omega_k)$.

It is wellknown that $\Phi(u; \omega_k) = \frac{1}{sn(u;\delta,\varepsilon)}$, where $\delta = -\frac{3}{2}e_k$, $\varepsilon = 3e_k^2 - \frac{g_2}{4}$.

Note that $\wp'(v) \to 0$ if and only if $v \to \omega_k$, $k = 1, 2, 3$.

Lemma 39. *We have*

$$\lim_{v\to\omega_k} \Psi(u) = \Phi(u; \omega_k) \exp\big(-\frac{\mu_1}{2}u + W(u)\big),$$

where $W(u) = \Big(\frac{1}{2}\frac{e_k-\wp'(w)}{6e_k^2-\frac{g_2}{2}}(\zeta(u + \omega_k) - \zeta(\omega_k - u) + 2e_k u)\Big)$.

Proof. We have

$$\lim_{v\to\omega_k} \frac{\ln\big(\sigma(u + v)\big) - \ln\big(\sigma(v - u)\big) - 2\zeta(v)u}{\wp'(v)} =$$
$$= \lim_{v\to\omega_k} \frac{\zeta(u + v) - \zeta(v - u) + 2\wp(v)u}{\wp''(v)} = \frac{\zeta(u + \omega_k) - \zeta(\omega_k - u) + 2e_k u}{6e_k^2 - \frac{g_2}{2}}$$

which is a meromorphic function without pole in zero.

Let $\hat\Psi$ be the function from the paper [15]

$$\hat\Psi(x; z, \eta) = \frac{\sigma(z + x + \eta)}{\sigma(z + \eta)\sigma(x)}\left[\frac{\sigma(z - \eta)}{\sigma(z + \eta)}\right]^{x/(2\eta)}.$$

Theorem 40.

$$\hat{\Psi}(u; z, \eta) = \Psi(u; v; -1)$$

$$\textit{for } v = z + \eta, \quad -\frac{\mu_1}{2} = \zeta(z+\eta) + \frac{1}{2\eta} \ln \left[\frac{\sigma(z-\eta)}{\sigma(z+\eta)} \right].$$

Proof. $\lim_{u\to 0} u\Psi(u) = 1 = \lim_{u\to 0} u\hat{\Psi}(u)$. The logarithm derivatives of this functions are equal for the given v, μ_1.

We will get the results similar to the Remark 31 for the function $\Psi(u)$ in our following works.

4. **The general elliptic genus.**

The general elliptic genus is the Hirzebruch genus L_f, where f is the exponential of the general elliptic formal group law $F_\mu(t_1, t_2)$.

Theorem 41. *(Integrality of the Hirzebruch genus.) The general elliptic formal group law $F_\mu(t_1, t_2)$ defines a 5-parametric family of E-integer Hirzebruch genera.*

Proof. The proof follows from the fact that the formal group of geomertic cobordisms (4) is universal and the general elliptic formal group law is defined over the ring $E = \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$.

Corollary 42. *Let $(\mu_4, \mu_6) = (0, 0)$. Then the corresponding formal group law is universal over the set of formal group laws over graduate rings A that are multiplicatively generated by a_k, $\deg a_k = -2k$, $k = 1, 2, 3$.*

Proof. The proof follows from the fact that the ring of coefficients of the formal group law generates all the ring.

Corollary 43. *Let $\delta = \mu_2$, $\varepsilon = \mu_2^2 - 4\mu_4$. The Hirzebruch genus L_f with the exponential $f(u) = sn(u)$ such that*

$$f'(u)^2 = 1 - 2\delta f(u)^2 + \varepsilon f(u)^4$$

defines an A-integer Hirzebruch genus, where $A = \mathbb{Z}[\mu_2, \mu_4]$.

5. **The general Krichever genus.**

5.1. **The Krichever genus.** Let $f_0(u) = \frac{1}{\Phi(u)}$, where $\Phi(u)$ is the Baker-Akhiezer function (51).

In the work [16] Krichever introduced the Hirzebruch genus defined by the function $f_0(u)$ and it was shown that it obtains the remarkable property of rigidity on SU-manifolds (Calabi–Yau manifolds) with the action of a circle S^1.

Let $a_2 = \wp(v)$, $a_3 = \wp'(v)$, $a_4 = \frac{1}{2}g_2$.

The Hirzebruch genus

$$L_{Kr} : \Omega_U \longrightarrow \mathbb{Q}[a_1, a_2, a_3, a_4],$$

defined by the series $f_{Kr}(u) = f_0(u)\exp(a_1 u)$, is called *the Krichever genus*. See [16].

Consider the transform
$$T\big(f(u)\big) = \frac{f(u)}{f'(u)}. \tag{68}$$
It brings Hurwitz series $f(u)$ such that $f(0) = 0,\ f'(0) = 1$ to Hurwitz series with the same property.

We have
$$\frac{d}{du} \ln f_{Kr}(u) = a_1 - \frac{1}{2} \frac{\wp'(u) + \wp'(v)}{\wp(u) - \wp(v)}. \tag{69}$$
Comparing the formulas (41) and (69) we obtain the following result:

Lemma 44. *The transform* (68) *brings f_{Kr} to the exponential of the ellipic formal group law, where $\wp'(v) = \wp'(w)$ and $a_1 = \frac{\mu_1}{2}$. Thus we obtain $\mu_1 = 2a_1,\ \mu_2 = 3a_2 - a_1^2$, $\mu_3 = -a_3,\ \mu_4 = 3a_2^2 - a_1a_3 - \frac{1}{2}a_4,\ \mu_6 = 0$.*

Remark 45. For $a_1 = \frac{\mu_1}{2}$ we have by definition $f_{Kr} = \frac{1}{\hat{\Phi}(u)}$.

Corollary 46. *Let $f(u)$ be the exponential of the elliptic formal group law $F_\mu(t_1, t_2)$ where $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, 0)$. Let $\wp(v) = \frac{1}{12}(4\mu_2 + \mu_1^2)$, $\wp'(v) = -\mu_3$. Then*
$$(\ln \hat{\Phi}(u))' = -\frac{1}{f(u)}.$$

For $a_1 = 0$ we obtain the result for the Baker-Akhiezer function (51):

Corollary 47. *Let $f(u)$ be the exponential of the elliptic formal group law $F_\mu(t_1, t_2)$ where $\mu = (0, \mu_2, \mu_3, \mu_4, 0)$. Let $\wp(v) = \frac{1}{3}\mu_2$, $\wp'(v) = -\mu_3$. Then*
$$\frac{\partial}{\partial u} \ln \Phi(u; v) = -\frac{1}{f(u)}.$$

5.2. **The general Krichever genus.** The Corollary 47 can be reformulated in the following way: The Baker-Akhiezer function $\Phi(u; v)$ for the proper v is a solution of the equation
$$\Phi'(u) + \frac{1}{f(u)} \Phi(u) = 0,$$
where $f(u)$ is the exponential of the elliptic formal group law with the parametra $\mu = (0, \mu_2, \mu_3, \mu_4, 0)$. By the Corollary 46 the function $\hat{\Phi}(u)$ is the solution of the same equation for the elliptic formal group law with the parametra $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, 0)$. The function $f_{Kr} = \frac{1}{\hat{\Phi}(u)}$ defines the Krichever genus. Let us define the general Krichever genus in the following way:

The general Krichever genus is the Hirzebruch genus L_ϕ, where $\phi(u) = \frac{1}{\Psi(u)}$, and $\Psi(u)$ the solution of the equation
$$\Psi' + \frac{1}{f(u)} \Psi = 0$$
such that $\lim\limits_{u \to 0} \big(\Psi(u) - \frac{1}{u}\big) =$ const, where $f(u)$ is the exponential of the general elliptic formal group law.

Comparing (64) and (41), we come to the following theorem:

Theorem 48. *The exponential of the formal group law corresponding to the general Krichever genus has the form*

$$\frac{1}{\Psi(u)} = \Phi(u;v)^{-1} \exp\left(\frac{\mu_1}{2}u\right)\left(\frac{\Phi(u;v)}{\Phi(u;-v)}\right)^{\frac{1}{2}(1-\alpha)}, \quad where$$

where $\wp(v) = \frac{1}{12}(4\mu_2 + \mu_1^2)$, $\wp'(w) = -\mu_3$ *and* $\alpha = \frac{\wp'(w)}{\wp'(v)}$.

Remark 49. The function $\Psi(u)$ concludes with the one defined by (63).

Corollary 50. *The general Krichever genus becomes the Krichever genus for* $v = w$.

Lemma 51. *In the vicinity of* $u = 0$ *the exponential of the general Krichever genus*

$$\frac{1}{\Psi(u)} = u + \sum \Psi_k \frac{u^{k+1}}{(k+1)!},$$

is a Hurwitz series over $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$, *where* $a_1 = \frac{\mu_1}{2}$, $a_2 = \wp(v) = \frac{1}{12}(4\mu_2 + \mu_1^2)$, $a_3 = \wp'(w) = -\mu_3$, $a_4 = \frac{1}{2}g_2(\mu)$, $a_6 = 4\mu_6 + \mu_3^2$.

Proof. We have

$$\frac{1}{\Psi(u)} = \sigma(u)\exp(\frac{\mu_1}{2}u)\exp(\psi(u,v)), \tag{70}$$

where

$$\psi(u,v) = \ln\sigma(v) - \frac{1}{2}(1-\alpha)\ln\sigma(v+u) - \frac{1}{2}(1+\alpha)\ln\sigma(v-u) - \alpha\zeta(v)u.$$

Teylor decomposition of the function $\psi(u,v)$ at $u = 0$ is given by

$$\psi(u,v) = \sum_{k=1}^{\infty} \frac{1}{2}((1-\alpha) + (-1)^{k-1}(1+\alpha))\wp^{(k-1)}(v)\frac{u^{k+1}}{(k+1)!}. \tag{71}$$

Using the Remark 34 and the notation $b_3 = \wp'(v)$ we obtain

$$\psi(u,v) = \sum_{n=0}^{\infty} r_{2n+2}(a_2, b_3^2, a_4)\frac{u^{2n+2}}{(2n+2)!} - (\alpha\, b_3)\sum_{n=0}^{\infty} q_{2n}(a_2, b_3^2, a_4)\frac{u^{2n+3}}{(2n+3)!}. \tag{72}$$

Thus $\psi(u,v)$ is a Hirwitz series over a_2, $a_6 = b_3^2$, a_4 and $a_3 = \alpha\, b_3$.
We have $\sigma(u) \in H\mathbb{Z}[\frac{g_2}{2}, 2g_3][[u]]$ and thus $\sigma(u) \in H\mathbb{Z}[a_2, b_3^2, a_4][[u]]$.
Thus $\frac{1}{\Psi(u)} \in H\mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$.

5.3. **The formal group law for the Krichever genus.**

The addition theorem, characterising the Krichever genus, was introduced in [7]. The universal properties of this genus were described in [3]. Unfortunately, the proof of theorem 6.23 of this work contains inaccuracies.

Following [17], we can write the addition theorem for the function $f_0(u) = 1/\Phi(u,v)$ in the form

$$f_0(u+v) = \frac{f_0(u)^2\frac{f_0(v)}{-f_0(-v)} - f_0(v)^2\frac{f_0(u)}{-f_0(-u)}}{f_0(u)f_0'(v) - f_0(v)f_0'(u)}. \tag{73}$$

Notice that if the function $f_0(u)$ gives a solution of the equation (73), then the product $\exp(cu)\, f_0(u)$ also gives a solution for any constant c.

The followig theorem characterises the Krichever genus f_{Kr} in terms of addition theorems. All the series considered in the theorem below are over $A \otimes \mathbb{Q}$.

Theorem 52. (A version of theorem 1 from [7]) *The function $f(u)$ such that $f(0) = 0,\ f'(0) = 1$ has an addition theorem of the form*

$$f(u+v) = \frac{f(u)^2 \xi_1(v) - f(v)^2 \xi_1(u)}{f(u)\xi_2(v) - f(v)\xi_2(u)} \tag{74}$$

(for some series $\xi_1(u)$ and $\xi_2(u)$ such that $\xi_1(0) = \xi_2(0) = 1$) if and only if $f(u) = f_{Kr}(u)$ is the Krichever genus.

We will give the proof in a few steps:

Remarks 53.

(1) Let $f(u)$ satisfy the equation (74) with some $\xi_1(u)$ and $\xi_2(u)$. Then the series $f(u)\exp(a_1 u)$ also satisfies the equation (74), where the series $\xi_1(u)$ and $\xi_2(u)$ are replaced by the series $\xi_1(u)\exp(2a_1 u)$ and $\xi_2(u)\exp(a_1 u)$.
(2) Let $f(u)$ satisfy the equation (74) with some $\xi_1(u)$ and $\xi_2(u)$. Then the same series $f(u)$ satisfies the equation (74), where the series $\xi_1(u)$ and $\xi_2(u)$ are replaced by the series $\xi_1(u) + \gamma_2 f(u)^2$ and $\xi_2(u) + \gamma_1 f(u)$.

Lemma 54. *Let the function $\widetilde{f}(u)$ such that $\widetilde{f}(0) = 0,\ \widetilde{f}'(0) = 1$ be a solution of the addition theorem of the form (74). Then $\widetilde{f}(u) = f(u)\exp(a_1 u)$, where $f(u)$ is the solution of the differential equation*

$$(f''' + 2a_2 f' - a_3 f)\, f - 3f'' f' = 0, \tag{75}$$

with initial conditions $f(0) = 0,\ f'(0) = 1,\ f''(0) = 0$. Here $a_1,\ a_2,\ a_3 \in A$.

Proof. Taking into account the Remarks 53, it is sufficient to prove the lemma with the following initial conditions:

$$f(0) = f''(0) = 0,\ f'(0) = 1; \quad \xi_1(0) = \xi_2(0) = 1,\ \xi_1''(0) = \xi_2'(0) = 0.$$

We have

$$f(u+v)\,[f(u)\xi_2(v) - f(v)\xi_2(u)] = f(u)^2\xi_1(v) - f(v)^2\xi_1(u). \tag{76}$$

Set $f(u) = f,\ \xi_1(u) = \xi_1,\ \xi_2(u) = \xi_2$. The series $f(v),\ \xi_1(v),\ \xi_2(v)$ up to v^4 have the following form:

$$f(u+v) \approx f + f'v + f''\frac{v^2}{2} + f'''\frac{v^3}{3!},$$

$$f(v) \approx v + f_2\frac{v^3}{3!}; \quad \xi_1(v) \approx 1 + \xi_{1,1}v + \xi_{1,3}\frac{v^3}{3!}; \quad \xi_2(v) \approx 1 + \xi_{2,2}\frac{v^2}{2} + \xi_{2,3}\frac{v^3}{3!}\,.$$

Substituing into (76):

$$\Big(f + f'v + f''\frac{v^2}{2} + f'''\frac{v^3}{3!}\Big)\left[f\Big(1 + \xi_{2,2}\frac{v^2}{2} + \xi_{2,3}\frac{v^3}{3!}\Big) - \Big(v + f_2\frac{v^2}{3!}\Big)\xi_2\right] =$$

$$= f\left[f + f'v + f''\frac{v^2}{2} + f'''\frac{v^3}{6} + \xi_{2,2}f\frac{v^2}{2} + \xi_{2,2}f'\frac{v^3}{2} + \xi_{2,3}f\frac{v^3}{3!}\right] - \Big(fv + f'v^2 + f''\frac{v^3}{2} + f_2 f\frac{v^3}{6}\Big)\xi_2 .$$

On the other hand

$$f^2\Big(1 + \xi_{1,1}v + \xi_{1,3}\frac{v^3}{6}\Big) - \Big(v + f_2\frac{v^2}{6}\Big)^2\xi_1 = f^2\Big(1 + \xi_{1,1}v + \xi_{1,3}\frac{v^3}{6}\Big) - \xi_1 v^2.$$

Comparing the coefficients at the corresponding degrees of v, we get:

At v we have

$$ff' - f\xi_2 = \xi_{1,1}f^2.$$

Therefore $\xi_2 = f' - \xi_{1,1}f$. We have: $\xi_2'(0) = f''(0) - \xi_{1,1}f'(0)$. Thus, if $f''(0) = \xi_2'(0) = 0$ and $f'(0) = 1$, then $\xi_{1,1} = 0$ and $\xi_2(u) = f'(u)$.

At v^2 we have

$$\frac{1}{2}ff'' + \frac{1}{2}\xi_{2,2}f^2 - f'\xi_2 = -\xi_1.$$

Using $\xi_2(u) = f'(u)$, we get:

$$\xi_1(u) = f'(u)^2 - \frac{1}{2}ff'' - \frac{1}{2}f_2f^2.$$

Thus, if $f''(0) = 0$ and $f'(0) = 1$, then the equation (74) is equivalent to the equation

$$f(u+v) = f(u)f'(v) + f(v)f'(u) - \frac{1}{2}f(u)f(v)\frac{f(u)f''(v) - f(v)f''(u)}{f(u)f'(v) - f(v)f'(u)} . \tag{77}$$

Set $\psi(u) = \big(\ln f(u)\big)'$. Then the equation (77) can be presented in the form

$$f(u+v) = \frac{1}{2}f(u)f(v)\left[\psi(u) + \psi(v) - \frac{\psi'(u) - \psi'(v)}{\psi(u) - \psi(v)}\right].$$

Thus,

$$f(u+v) = \frac{1}{2}f(u)f(v)\left[\psi(u) + \psi(v) - \partial_+ \ln\big(\psi(u) - \psi(v)\big)\right]. \tag{78}$$

At v^3 we have

$$\frac{1}{6}ff''' + \frac{1}{2}\xi_{2,2}ff' + \frac{1}{6}\xi_{2,3}f^2 - \frac{1}{2}f''\xi_2 - \frac{1}{6}f_2f\xi_2 = \frac{1}{6}\xi_{1,3}f^2.$$

Because $\xi_2(u) = f'(u)$ and thus $\xi_{2,2} = f_2$, we get

$$[f''' + 2f_2f' + (\xi_{2,3} - \xi_{1,3})f]\, f - 3f''f' = 0.$$

Setting $a_2 = f_2$ and $a_3 = \xi_{2,3} - \xi_{1,3}$, we get the proof of the lemma.

Corollary 55. *Let the function $\widetilde{f}(u)$ such that $\widetilde{f}(0) = 0,\ \widetilde{f}'(0) = 1$ be a solution of the addition theorem of the form (74). Then it has an addition theorem of the form*

$$f(u+v) = \frac{1}{2}f(u)f(v)\left[\psi(u) + \psi(v) - \frac{\psi'(u) - \psi'(v)}{\psi(u) - \psi(v)}\right] \tag{79}$$

where $\psi(u) = \frac{f'(u)}{f(u)} = \frac{1}{u} + \psi_0(u)$, $\psi_0(u) \in A \otimes \mathbb{Q}[[u]]$, *and* $\psi(u)$ *satisfies the differential equation*

$$(\psi')^2 = \psi^4 - 2a_2\psi^2 + a_3\psi - a_4. \tag{80}$$

Proof. Using the formulae

$$\frac{f''}{f} = \psi' + \psi^2, \quad \frac{f'''}{f} = \psi'' + 3\psi'\psi + \psi^3,$$

we get from (75) the equation

$$(\psi'' + 3\psi'\psi + \psi^3 + 2a\psi - a_3) - 3(\psi' + \psi^2)\psi = 0.$$

Thus

$$\psi'' - 2\psi^3 + 2a_2\psi - a_3 = 0. \tag{81}$$

Multiplying the equation (81) by $2\psi'$ and integrating, we come to the equation

$$(\psi')^2 = \psi^4 - 2a_2\psi^2 + a_3\psi - a_4.$$

Last step of the proof of theorem 52. Consider the function

$$f(u) = \frac{1}{\Phi(u;w)} = \frac{\sigma(u)\sigma(w)}{\sigma(w-u)} \exp(-\zeta(w)u).$$

We have $f(u) = \sigma(u) \exp \psi(u,w)$ with $\psi(0,w) = 0$, $\psi'(0,w) = 0$.
Thus $f(0) = f''(0) = 0$, $f'(0) = 1$.

The given function has an addition theorem (73) of the form (74), where $\xi_1(u) = -\frac{f(u)}{f(-u)}$, $\xi_2(u) = f'(u)$. Thus the function $f(u)$ is the solution of the equation (75) with the given initial conditions. Using the uniqueness of the solution of this equation, we get the proof.

Remark 56. In the proof of the lemma 54 we have obtained the formula

$$\xi_1(u) = (f')^2 - \frac{1}{2}ff'' - \frac{1}{2}f_2f^2.$$

Thus, for the given function $f(u)$ we have

$$f(-u) = -\frac{f(u)}{(f'(u))^2 - \frac{1}{2}f(u)f''(u) - \frac{1}{2}f_2f(u)^2}.$$

Therefore in the case of an odd function $f(u)$ we get the equation

$$(f')^2 - \frac{1}{2}ff'' - \frac{1}{2}f_2f^2 = 1 \quad \text{with the initial conditions} \quad f(0) = 0,\ f'(0) = 1,$$

and the solution $f(u) = sn(u)$.

6. Appendix and Applications.

The following examples are related to the authomorphisms of the elliptic formal group laws.

Example 57. In the notations of section 2.9, let $n = 2$ and $\alpha = -1$. Then $\mu_1 = \mu_3 = 0$.
In this case we obtain an elliptic curve in Tate coordinates (8)

$$s = t^3 + \mu_2 t^2 s + \mu_4 t s^2 + \mu_6 s^3.$$

Set $s = tv$. Then

$$\frac{v}{1 + \mu_2 v + \mu_4 v^2 + \mu_6 v^3} = \tau$$

where $\tau = t^2$. Set $1 + \mu_2 z + \mu_4 z^2 + \mu_6 z^3 = (1 + \gamma_1 z)(1 + \gamma_2 z)(1 + \gamma_3 z)$. Using the classical Lagrange inversion formula, we obtain

$$v(\tau) = -\frac{1}{2\pi i} \oint\limits_{|z|=\varepsilon} \ln \Big[1 - \frac{\tau}{z}(1 + \gamma_1 z)(1 + \gamma_2 z)(1 + \gamma_3 z)\Big]\, dz =$$

$$= \sum_{n=1}^{\infty} \frac{\tau^n}{n} \frac{1}{2\pi i} \oint\limits_{|z|=\varepsilon} \frac{(1 + \gamma_1 z)^n (1 + \gamma_2 z)^n (1 + \gamma_3 z)^n}{z^n}\, dz =$$

$$= \sum_{n \geqslant 1} \sum_{j_1 + j_2 + j_3 = n-1} \binom{n}{j_1} \binom{n}{j_2} \binom{n}{j_3} \gamma_1^{j_1} \gamma_2^{j_2} \gamma_3^{j_3} \frac{\tau^n}{n}.$$

Therefore we have

$$s(t) = t \sum_{n \geqslant 1} \sum_{j_1 + j_2 + j_3 = n-1} \binom{n}{j_1} \binom{n}{j_2} \binom{n}{j_3} \gamma_1^{j_1} \gamma_2^{j_2} \gamma_3^{j_3} \frac{t^{2n}}{n} \tag{82}$$

where $\gamma_1 + \gamma_2 + \gamma_3 = \mu_2$, $\gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3 = \mu_4$, $\gamma_1\gamma_2\gamma_3 = \mu_6$.

Formula (21) gives the elliptic formal group law

$$F_\mu(t_1, t_2) = t_1 + t_2 + b \frac{(\mu_2 + 2\mu_4 m + 3\mu_6 m^2)}{(1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3)}.$$

Formula (24) gives the equation on the exponential

$$\mu_6 f'(u)^3 + (3\mu_6 + (\mu_4^2 - 3\mu_2\mu_6) f(u)^2) f'(u)^2 + (27\mu_6^2 - \mu_2^2\mu_4^2 + 4\mu_2^3\mu_6 - 18\mu_2\mu_4\mu_6 + 4\mu_4^3) f(u)^6 +$$
$$+ (18\mu_4\mu_6 - 12\mu_2^2\mu_6 + 2\mu_2\mu_4^2) f(u)^4 + (12\mu_2\mu_6 - \mu_4^2) f(u)^2 - 4\mu_6 = 0. \tag{83}$$

Lemma 58. *The exponential $f(u)$ of an elliptic formal group law is an odd function if and only if $\mu_1 = \mu_3 = 0$.*

Proof. Let $\mu_1 = \mu_3 = 0$. Then it follows form (8) that $s(t)$ is an odd function of t, thus $b(t, -t) = 0$. It follows from (21) that $F_\mu(t, -t) = 0$, thus $g(t) = -g(-t)$ and $f(t) = -f(-t)$. The inverse follows from (2) and the series decomposition of the right part of (22).

Corollary 59. *In the considered case the exponential* $f(u) \in \mathbb{Z}_{(2)}[\mu_2, \mu_4, \mu_6][[u]]$ *and the logarithm* $g(t) \in \mathbb{Z}_{(2)}[\mu_2, \mu_4, \mu_6][[t]]$*. Thus the exponential gives a strong isomorphism over* $\mathbb{Z}_{(2)}[\mu_2, \mu_4, \mu_6]$ *of the formal group law* $F_\mu(t_1, t_2)$ *given by (*21*) and the linear group* L*.*

Example 60. Let $n = 3$ and let α be a root of 1 of order 3. Then we obtain *the equianharmonic case* $\mu_1 = \mu_2 = \mu_4 = 0$.

In this case we obtain an elliptic curve in Tate coordinates (8)

$$s = t^3 + \mu_3 s^2 + \mu_6 s^3 \quad \text{with} \quad \Delta = -27(4\mu_6 + \mu_3^2)^2.$$

The elliptic formal group law (20) takes the form

$$F_\mu(t_1, t_2) = \frac{(t_1 + t_2)(1 + \mu_6 m^3) + \mu_3 m^2 + 3\mu_6 m^2 b}{(1 + \mu_6 m^3)(1 - \mu_3 b) - \frac{bm}{p}\mu_3(1 - \mu_3 b - \mu_6 b^2)}.$$

Formula (24) gives an equation on its exponential $f(u)$:

$$\mu_6 \left[f'(u)^3 + 3f'(u)^2 + 27\mu_6 f(u)^6 + 18\mu_3 f(u)^3 - 4\right] = -\mu_3^2 \left[f'(u)^2 + 4\mu_3 f(u)^3 - 1\right]. \tag{84}$$

Example 61. Let $n = 4$ and let α be a root of 1 of order 4. Then we obtain *the Lemniscate case* $\mu_1 = \mu_2 = \mu_3 = \mu_6 = 0$.

In this case we obtain an elliptic curve in Tate coordinates (8)

$$s = t^3 + \mu_4 t s^2, \quad \text{with} \quad \Delta = -64\mu_4^3.$$

Let $v = \mu_4 t s$ and $\tau = \mu_4 t^4$. The equation becomes

$$v^2 - v + \tau = 0, \quad \text{thus} \quad v(\tau) = \frac{1}{2}(1 - \sqrt{1 - 4\tau}).$$

Formula (32) gives the elliptic formal group law

$$F_\mu(t_1, t_2) = t_1 + t_2 + \frac{2\mu_4 bm}{1 + \mu_4 m^2}.$$

Formula (24) gives an equation on its exponential $f(u)$:

$$f(u)' = \sqrt{1 - 4\mu_4 f(u)^4}.$$

It implies from (34) that the function $f(u) = \frac{-2\wp(u; 4\mu_4, 0)}{\wp'(u; 4\mu_4, 0)}$ is a solution of this equation and the exponential of the formal group F_μ.

Notice that we can choose the half-periods ω_1 and ω_2 of the function $\wp(u; 4\mu_4, 0)$ such that $\omega_2 = i\omega_1$.

Now let us use equation (3) to find the image of $\mathcal{A}_{(2)} \to \mathbb{Z}[\mu_4]$. We have

$$F_\mu(t, t) = 2t\frac{1 - 2\mu_4 ts}{1 + 4\mu_4 t^4}, \qquad \frac{\partial}{\partial t_2} F_\mu(t, t_2)|_{t_2 = 0} = \frac{2t^3}{s} - 1.$$

The function $Cat(\tau) = \frac{1}{2\tau}(1 - \sqrt{1-4\tau})$ is the generating function of the Catalan numbers, i.e. $Cat(\tau) = \sum_{n\geqslant 0} C_n \tau^n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$. Using our notations we obtain $v(\tau) = \tau Cat(\tau)$, and $s = t^3 Cat(\mu_4 t^4)$. Thus

$$s(t) = \sum_{n\geqslant 0} C_n \mu_4^n t^{4n+3}.$$

On the other hand it follows from (82) that $C_n = \frac{1}{n+1}\binom{2n}{n} = (-1)^n \frac{1}{2n+1} \sum_{j=0}^{2n} (-1)^j \binom{2n+1}{j}\binom{2n+1}{j+1}$.

Let a_i for $i = 1, 2, 3, \ldots$ be the generators of $\mathcal{A}_{(2)}$. For $\phi : \mathcal{A}_{(2)} \to \mathbb{Z}[\mu_4]$ we have $\phi(a_i) = 0$ for $i \neq 4k$, $k = 1, 2, 3, \ldots$. Thus

$$\frac{\partial}{\partial t_2} F_\mu(t, t_2)|_{t_2=0} = 1 + \sum_{k\geqslant 1} \phi(a_{4k}) t^{4k}.$$

Summarizing this formulas, we get

$$(2 + \sum_{k\geqslant 1} \phi(a_{4k}) t^{4k})(\sum_{n\geqslant 0} C_n \mu_4^n t^{4n}) = 2.$$

Thus $\phi(a_{4k})$ are given by the system of formulas

$$\sum_{q=0}^{m} \phi(a_{4(m-q)}) C_q \mu_4^q = 0, \quad \text{where} \quad \phi(a_0) = 2, \quad m \geq 1.$$

For $m = 1$ we get $\phi(a_4) = -2\mu_4$. We get the relations between $\phi(a_{4q})$:

$$\sum_{q=0}^{m} (-1)^q 2^{m-q-r(m)} C_q \phi(a_{4(m-q)}) \phi(a_4)^q = 0, \quad r(m) = \gcd(2C_m, 2^{m-q} C_q), \quad m \geq 1.$$

Thus the image of $\mathcal{A}_{(2)} \to \mathbb{Z}[\mu_4]$ the generators can be chosen as $\alpha_m = \phi(a_{4m})$, and the image will be

$$\mathbb{Z}[\alpha_m]/J$$

where J is generated by $(-1)^m 2^{1-r(m)} C_m \alpha_1^m + \sum_{q=1}^{m} (-1)^{m-q} 2^{q-r(m)} C_{m-q} \alpha_q \alpha_1^{m-q}$.

Explicitly, this equations for $m = 2, 3, 4$ will be $2\alpha_2 = -\alpha_1^2$, $4\alpha_3 = 3\alpha_1^3 + 2\alpha_2\alpha_1 = 2\alpha_1^3$, and $8\alpha_4 = -9\alpha_1^4 - 4\alpha_2\alpha_1^2 + 4\alpha_3\alpha_1 = -5\alpha_1^4$.

Example 62. Let $n = 4$ and let α be a root of 1 of order 6. Then $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$.

In this case we obtain an elliptic curve in Tate coordinates (8)

$$s = t^3 + \mu_6 s^3, \quad \text{with} \quad \Delta = -432\mu_6^2.$$

Formula (32) gives the elliptic formal group law

$$F_\mu(t_1, t_2) = t_1 + t_2 + \frac{3\mu_6 b m^2}{1 + \mu_6 m^3}.$$

Formula (24) gives a differential equation on its exponential $f(u)$:

$$27\mu_6 f(u)^6 = (1 - f'(u))(2 + f'(u))^2.$$

It implies from (34) that the function $f(u) = \frac{-2\wp(u;0,4\mu_6)}{\wp'(u;0,4\mu_6)}$ is a solution of this equation and the exponential of the formal group.

Notice that we can choose the half-periods of the function $\wp(u; 0, 4\mu_6)$ ω_1 and ω_2 such that $\omega_2 = \frac{1+i\sqrt{3}}{2}\omega_1$.

Now let us consider some examples that are not related to the authomorphisms of the elliptic formal group laws directly.

Example 63. Let $\mu_3 = \mu_4 = \mu_6 = 0$.

In this case we obtain an elliptic curve in Tate coordinates (8)

$$(1 - \mu_1 t - \mu_2 t^2)s = t^3 \quad \text{with} \quad \Delta = 0.$$

The elliptic formal group law (19) takes the form

$$F_\mu(t_1, t_2) = \frac{t_1 + t_2 - \mu_1 t_1 t_2}{1 + \mu_2 t_1 t_2}.$$

The exponential is a solution of the equation (25):

$$f(u)' = 1 - \mu_1 f(u) - \mu_2 f(u)^2.$$

Let $\alpha + \beta = \mu_1$, $\alpha\beta = -\mu_2$. The solution of this equation is

$$f(u) = \frac{e^{\alpha u} - e^{\beta u}}{\alpha e^{\alpha u} - \beta e^{\beta u}}.$$

We have (see [2]):

$$\frac{e^{\alpha u} - e^{\beta u}}{\alpha e^{\alpha u} - \beta e^{\beta u}} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^n A_{n,k}\alpha^n\beta^{n-k}\frac{u^{n+1}}{(n+1)!}.$$

Here $A_{n,k}$ are defined in the following way: a transposition $i_1, ...i_{n+1}$ of numbers $1, ..., n+1$ is said to have a decrease at l if $i_l > i_{l+1}$. Then $A_{n,k}$ is the number of transpositions of $n+1$ numbers, having k decreases.

So $f_n = \sum_{k=0}^{n} A_{n,k}\alpha^n\beta^{n-k}\frac{1}{(n+1)!}$.

On the other hand, we have $g_2 = \frac{1}{12}(\mu_1^2 + 4\mu_2)^2$, $g_3 = -\frac{1}{6^3}(\mu_1^2 + 4\mu_2)^3$, so formula (40) gives the exponential of the formal group law F_μ:

$$f(u) = -2\frac{\wp(u) - \frac{1}{12}(\mu_1^2 + 4\mu_2)}{\wp'(u) - \mu_1\wp(u) + \frac{1}{12}\mu_1(\mu_1^2 + 4\mu_2)}, \tag{85}$$

where $\wp(u) = \wp(u; g_2, g_3)$. It implies that $f(u)$ gives a solution of the equation (25).

Remark 64. The $\wp$-function for $\Delta = 0$.

We see that in the considered case $\Delta = g_2^3 - 27g_3^2 = 0$. Set $g_2 = \frac{4}{3}\gamma^4$, $g_3 = -\frac{8}{27}\gamma^6$. In this case (see [1])

$$\sigma(z, g_2, g_3) = \frac{1}{2\gamma}e^{-\frac{1}{6}\gamma^2 z^2}(e^{\gamma z} - e^{-\gamma z}),$$

thus

$$\wp(z, g_2, g_3) = -\zeta' = -\frac{2}{3}\gamma^2 + \gamma^2 cth(\gamma z)^2.$$

It follows from formula (85) that in the considered case

$$f(u) = \frac{1}{\gamma cth(\gamma z) + \frac{1}{2}\mu_1} \text{ for } \gamma = \frac{1}{2}\sqrt{\mu_1^2 + 4\mu_2}.$$

Corollary 65. *In the case $\mu_2 = \mu_3 = \mu_4 = \mu_6 = 0$ we get*

$$F_\mu(t_1, t_2) = t_1 + t_2 - \mu_1 t_1 t_2,$$

so F_μ is the multiplicative formal group.

The exponential satisfies the equation

$$f(u)' = 1 - \mu_1 f(u).$$

Using the condition $f(u) = u + ...$, we get

$$\mu_1 f(u) = 1 - e^{-\mu_1 u}. \tag{86}$$

On the other hand, (40) gives

$$f(u) = -2\frac{\wp(u; \frac{1}{12}\mu_1^4, -\frac{1}{6^3}\mu_1^6) - \frac{1}{12}\mu_1^2}{\wp'(u; \frac{1}{12}\mu_1^4, -\frac{1}{6^3}\mu_1^6) - \mu_1\wp(u; \frac{1}{12}\mu_1^4, -\frac{1}{6^3}\mu_1^6) + \frac{1}{12}\mu_1^3}.$$

Using remark 64 we see that this formula for $f(u)$ conclides with (86).

It follows from (86) that

$$f(u) = u + \sum(-1)^n \mu_1^n \frac{u^{n+1}}{(n+1)!}.$$

Thus B_f is generated by $f_n = \frac{(-1)^n\mu_1^n}{(n+1)!}$, and

$$B_f = \mathbb{Z}[b_n]/J$$

where J is an ideal generated by polynomials $(n+1)!b_n - 2^n b_1^n$.

Corollary 66. *In the case $\mu_1 = \mu_3 = \mu_4 = \mu_6 = 0$ we get*

$$F_\mu(t_1, t_2) = \frac{t_1 + t_2}{1 + \mu_2 t_1 t_2},$$

which is a formal group coming from the addition formula for the hyperbolic tangent function.

The exponential satisfies the equation

$$f(u)' = 1 - \mu_2 f(u)^2.$$

Using the condition $f(u) = u + ...$, we get

$$f(u) = \frac{1}{\sqrt{\mu_2}} th(\sqrt{\mu_2}u).$$

Thus

$$f(u) = \sum_{k=1}^{\infty} 2^{2k}(2^{2k} - 1)B_{2k}\mu_2^{k-1}\frac{u^{2k-1}}{(2k)!},$$

where B_{2k} denotes the Bernoulli numbers. It follows that B_f is generated by $f_{2k-2} = \frac{2^{2k}(2^{2k}-1)B_{2k}\mu_2^{k-1}}{(2k)!}$ and we have $f_{2k-1} = 0$. Thus $f_0 = 6B_2 = 1$, $f_2 = 10B_4\mu_2 = -\frac{1}{3}\mu_2$.

$$B_f = \mathbb{Z}[b_{2k}]/J$$

where J is an ideal generated by the polynomials $(2k)!b_{2k-2} - 2^{2k}(2^{2k}-1)B_{2k}(-3b_2)^{k-1}$.

Example 67. Let $\mu_1 = \mu_2 = \mu_4 = \mu_6 = 0$.

In this case we obtain an elliptic curve in Tate coordinates (8)

$$s = t^3 + \mu_3 s^2, \quad \text{with} \quad \Delta = -27\mu_3^4.$$

The elliptic formal group law (19) takes the form

$$F_\mu(t_1, t_2) = \frac{(t_1 + t_2) - \mu_3(t_1 + t_2)b - \mu_3 t_1 t_2 m}{(1 - \mu_3 b)^2}. \tag{87}$$

Thus

$$F_\mu(t_1, t_2) = (t_1 + t_2) - \mu_3 t_1 t_2(2t_1^2 + 3t_1t_2 + 2t_2^2) + O(t^7).$$

Though $\gcd(2, 3) = 1$, we obtain the homomorphism $\mathcal{A} \to \mathbb{Z}[\mu_3]$ classifying the formal group law (87) to be an epimorphism, that is there exists a set of generators $\{a_n\}$ in $\mathcal{A}$, such that $\phi(a_n) = 0$ for $n \neq 3$, $\phi(a_3) = \mu_3$.

Formula (24) gives an equation on the exponential $f(u)$ of the formal group law F_μ:

$$f'(u)^2 = -4\mu_3 f(u)^3 + 1. \tag{88}$$

Therefore, we get

$$f(u) = \frac{-1}{\mu_3}\wp(u + c; 0, -\mu_3^2),$$

where c is given by the conditions

$$\wp(c; 0, -\mu_3^2) = 0, \quad \wp'(c; 0, -\mu_3^2) = -\mu_3.$$

On the other hand, we have $g_2 = 0$, $g_3 = -\mu_3^2$, so the function

$$f(u) = -2\frac{\wp(u; 0, -\mu_3^2)}{\wp'(u; 0, -\mu_3^2) - \mu_3}$$

is the exponential of the formal group F_μ. It implies that

$$\frac{-1}{\mu_3}\wp(u + c; 0, -\mu_3^2) = -2\frac{\wp(u; g_2, g_3)}{\wp'(u; g_2, g_3) - \mu_3}.$$

Example 68. Let $\mu_3^2 = -3\mu_6$, $2\mu_3\mu_4 = 3\mu_1\mu_6$, $\mu_4^2 = 3\mu_2\mu_6$, $\mu_6 \neq 0$.

In this case we obtain an elliptic curve in Tate coordinates (8)

$$s = t^3 + \mu_1 ts + \mu_2 t^2 s + \mu_3 s^2 + \mu_4 ts^2 + \mu_6 s^3.$$

Formula (24) gives an equation on the exponential $f(u)$ of the formal group law:

$$f'(u)^3 = ((1 - \frac{1}{2}\mu_1 f(u))^3 - 3\mu_3 f(u)^3)^2. \tag{89}$$

We have $g_2 = 0$, $\quad g_3 = -\mu_6 = \frac{1}{3}\mu_3^2$, $\quad \Delta = g_2^3 - 27g_3^2 = -3\mu_3^4$.

It follows from (40) that the solution of (89) is

$$f(u) = -2\frac{\wp(u;0,\frac{1}{3}\mu_3^2)}{\wp'(u,0,\frac{1}{3}\mu_3^2) - \mu_1\wp(u;0,\frac{1}{3}\mu_3^2) - \mu_3}.$$

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